ON THE AVERAGE BOX DIMENSIONS OF GRAPHS OF TYPICAL CONTINUOUS FUNCTIONS

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Abstract. Let X be a bounded subset of $\mathbb{R}^d$ and write $C_{\mathsf{u}}(X)$ for the set of uniformly continuous functions on X equipped with the uniform norm. The lower and upper box dimensions, denoted by $\underline{\dim}_{\mathsf{B}}(\text{graph}(f))$ and $\overline{\dim}_{\mathsf{B}}(\text{graph}(f))$, of the graph graph$(f)$ of a function $f \in C_{\mathsf{u}}(X)$ are defined by

$$\underline{\dim}_{\mathsf{B}}(\text{graph}(f)) = \liminf_{\delta \searrow 0} \frac{\log N_\delta(\text{graph}(f))}{-\log \delta},$$
$$\overline{\dim}_{\mathsf{B}}(\text{graph}(f)) = \limsup_{\delta \searrow 0} \frac{\log N_\delta(\text{graph}(f))}{-\log \delta},$$

where $N_\delta(\text{graph}(f))$ denotes the number of δ-mesh cubes that intersect graph(f).

Hyde et al have recently proved that the box counting function

$$\frac{\log N_\delta(\text{graph}(f))}{-\log \delta} \tag{$*$}$$

of the graph of a typical function $f \in C_{\mathsf{u}}(X)$ diverges in the worst possible way as $\delta \searrow 0$. More precisely, Hyde et al showed that for a typical function $f \in C_{\mathsf{u}}(X)$, the lower box dimension of the graph of f is as small as possible and if X has only finitely many isolated points, then the upper box dimension of the graph of f is as big as possible.

In this paper we will prove that the box counting function $(*)$ of the graph of a typical function $f \in C_{\mathsf{u}}(X)$ is spectacularly more irregular than suggested by the result due to Hyde et al. Namely, we show the following surprising result: not only is the box counting function in $(*)$ divergent as $\delta \searrow 0$, but it is so irregular that it remains spectacularly divergent as $\delta \searrow 0$ even after being "averaged" or "smoothened out" using exceptionally powerful averaging methods including *all* higher order Hölder and Cesaro averages and *all* higher order Riesz-Hardy logarithmic averages. For example, if the box dimension of X exists, then we show that for a typical function $f \in C_{\mathsf{u}}(X)$, *all* the higher order lower Hölder and Cesaro averages of the box counting function $(*)$ are as small as possible, namely, equal to the box dimension of X, and if, in addition, X has only finitely many isolated points, then *all* the higher order upper Hölder and Cesaro averages of the box counting function $(*)$ are as big as possible, namely, equal to the box dimension of X plus 1.

1. Statements of the Main Results.

1.1. Introduction. Recall that in a metric space $\mathcal{X}$, a set E is called co-meagre if its complement is meagre, and we say that a typical element $x \in \mathcal{X}$ has property P if the set $E = \{x \in \mathcal{X} \,|\, x \text{ has property } \mathsf{P}\}$ is co-meagre, see Oxtoby [Ox] for more details.

For a bounded subset X of $\mathbb{R}^d$, we let $C_{\mathsf{u}}(X)$ denote the set of uniformly continuous functions on X. It is well-known, and easy to see, that a uniformly continuous function $f : X \to \mathbb{R}$ on a bounded subset X of $\mathbb{R}^d$ is bounded, and the space $C_{\mathsf{u}}(X)$ of uniformly continuous functions on X can be equipped with the uniform norm $\|\cdot\|_\infty$ to form a normed space $(C_{\mathsf{u}}(X), \|\cdot\|_\infty)$. It is well-known that the normed space $(C_{\mathsf{u}}(X), \|\cdot\|_\infty)$ is a Banach space, and below we will always equip $C_{\mathsf{u}}(X)$ with the uniform norm. We emphasise that the set X, except from being bounded, is completely arbitrary; for example, we are not assuming that X is compact or Borel. Hyde et al [HyLaOlPeSh] have recently investigated the lower and the upper box dimension of the graph of a typical (in the sense of Baire) function $f \in C_{\mathsf{u}}(X)$. More precisely, Hyde et al [HyLaOlPeSh] proved that for a typical function $f \in C_{\mathsf{u}}(X)$, the lower box dimension of the graph of f is as small as possible, namely, equal to the lower box dimension of X, and if X has only finitely many isolated points, then the upper box dimension of the graph of f is as big as possible, namely, equal to the upper box dimension of X plus 1, see Theorem A below. The Hausdorff and packing dimensions of the graph of a typical continuous function $f : [0,1] \to \mathbb{R}$ have also been studied by Mauldin & Williams [MaWi] and Humke & Petruska [HuPe], respectively. Indeed, Mauldin & Williams [MaWi] proved that the Hausdorff dimension of the graph of a typical continuous function $f : [0,1] \to \mathbb{R}$ is as small as possible, namely, equal to 1, and Humke & Petruska [HuPe] proved that the packing dimension of the graph of a typical continuous function $f : [0,1] \to \mathbb{R}$ is as big as possible, namely, equal to 2. The purpose of this paper is to study this dichotomy in more detail. More precisely, we prove that the box dimension of the graph of a typical function $f \in C_{\mathsf{u}}(X)$ is spectacularly more irregular than suggested by the results in [HuPe,HyLaOlPeSh,MaWi]. However, we first recall that the box dimensions of a subset E of Euclidean space is defined as the lower and upper limits of the box counting function $\frac{\log N_\delta(E)}{-\log \delta}$ as $\delta \searrow 0$ where $N_\delta(E)$ denotes the number of δ-mesh cubes that intersect E (the precise definitions will be given below). We can now state an informal version of your main result. This result says, somewhat surprisingly, that the box counting function $\frac{\log N_\delta(\mathrm{graph}(f))}{-\log\delta}$ of the graph graph(f) of a typical function $f \in C_{\mathsf{u}}(X)$ is dramatically more irregular than suggested by the results in [HuPe,HyLaOlPeSh,MaWi].

Informal version of Theorems 1.1, 2.1 and 3.1. *The box counting function*

$$\Lambda_f(\delta) = \frac{\log N_\delta(\mathrm{graph}(f))}{-\log\delta} \tag{1.1}$$

of the graph graph(f) *of a typical function* $f \in C_{\mathsf{u}}(X)$ *is so irregular that it remains spectacularly divergent as* $\delta \searrow 0$ *even after being "averaged" using very general and powerful averaging methods including higher order Hölder and Cesaro averages and higher order Riesz-Hardy logarithmic averages. For example, if we define the n'th order Hölder averages, denoted by* $\Lambda_f^n(t)$*, of the box counting function in (1.1) inductively by*

$$\Lambda_f^0(t) = \Lambda_f(e^{-t}),$$
$$\Lambda_f^n(t) = \frac{1}{t}\int_1^t \Lambda_f^{n-1}(s)\,ds,$$

for $n \in \mathbb{N}$ *and* $f \in C_{\mathsf{u}}([0,1]^d)$*, then a typical continuous function* $f : [0,1]^d \to \mathbb{R}$ *satisfies*

$$\liminf_{t\to\infty} \Lambda_f^n(t) = d,$$
$$\limsup_{t\to\infty} \Lambda_f^n(t) = d+1,$$

for all $n \in \mathbb{N}\cup\{0\}$*.*

1.2. Statement of the main results. We start by recalling the definition of the lower and upper box dimensions of subsets of $\mathbb{R}^m$. For $\delta > 0$, let

$$\mathcal{Q}_\delta^m = \left\{ \prod_{i=1}^m [n_i\delta, (n_i+1)\delta] \,\middle|\, n_1, \ldots, n_m \in \mathbb{Z} \right\} \tag{1.2}$$

denote the standard δ-grid in $\mathbb{R}^m$, and for a bounded subset E of $\mathbb{R}^m$ write

$$N_\delta(E) = \left| \left\{ Q \in \mathcal{Q}_\delta^m \,\middle|\, Q \cap E \neq \varnothing \right\} \right| \tag{1.3}$$

for the number of cubes in $\mathcal{Q}_\delta^m$ that intersect E. The lower and upper box dimensions of E are now defined by

$$\underline{\dim}_{\mathsf{B}}(E) = \liminf_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta}, \tag{1.4}$$

and

$$\overline{\dim}_{\mathsf{B}}(E) = \limsup_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta}, \tag{1.5}$$

respectively. If the lower and upper box dimensions of E coincide, then we will say that the box dimension of E exists, and we will denote the common value by $\dim_{\mathsf{B}}(E)$, i.e. if $\underline{\dim}_{\mathsf{B}}(E) = \overline{\dim}_{\mathsf{B}}(E)$, then we will write

$$\dim_{\mathsf{B}}(E) = \underline{\dim}_{\mathsf{B}}(E) = \overline{\dim}_{\mathsf{B}}(E).$$

The reader is referred to Falconer [Fa] for a thorough discussion of the properties of the box dimensions.

For $f \in C_{\mathsf{u}}(X)$, we will write graph(f) for the graph of f, i.e.

$$\mathrm{graph}(f) = \left\{ (x, f(x)) \,\middle|\, x \in X \right\}.$$

In [HyLaOlPeSh] the authors found the box dimensions of the graphs of typical functions in $C_{\mathsf{u}}(X)$; this is the contents of Theorem A below.

Theorem A [HyLaOlPeSh]. *Let X be a bounded subset of $\mathbb{R}$.*

(1) *For all $f \in C_{\mathsf{u}}(X)$, we have*

$$\begin{aligned}\underline{\dim}_{\mathsf{B}}(X) &\le \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(f))\\ &\le \overline{\dim}_{\mathsf{B}}(\mathrm{graph}(f))\\ &\le \sup_{\varphi\in C_{\mathsf{u}}(X)} \overline{\dim}_{\mathsf{B}}(\mathrm{graph}(\varphi)) \le \overline{\dim}_{\mathsf{B}}(X)+1\,.\end{aligned}$$

(2) *For a typical function $f \in C_{\mathsf{u}}(X)$, we have*

$$\underline{\dim}_{\mathsf{B}}(\mathrm{graph}(f)) = \underline{\dim}_{\mathsf{B}}(X)\,.$$

(3) (i) *For a typical function $f \in C_{\mathsf{u}}(X)$, we have*

$$\overline{\dim}_{\mathsf{B}}(\mathrm{graph}(f)) = \sup_{\varphi\in C_{\mathsf{u}}(X)} \overline{\dim}_{\mathsf{B}}(\mathrm{graph}(\varphi)) \le \overline{\dim}_{\mathsf{B}}(X)+1\,.$$

(ii) *If, in addition, X only has finitely many isolated points, then for a typical function $f \in C_{\mathsf{u}}(X)$, we have*

$$\overline{\dim}_{\mathsf{B}}(\mathrm{graph}(f)) = \overline{\dim}_{\mathsf{B}}(X)+1\,.$$

Theorem A says that for a typical $f \in C_{\mathsf{u}}(X)$, the lower and upper box dimensions are as big and as small as they can be, respectively. In order to analyse this dichotomy in more detail, we introduce the following notation. Namely, for a bounded subset E of $\mathbb{R}^d$, we define the box counting function $\Delta_E : (0,\infty) \to [0,\infty]$ of E by

$$\Delta_E(t) = \frac{\log N_{e^{-t}}(E)}{-\log e^{-t}} = \frac{\log N_{e^{-t}}(E)}{t}\,. \tag{1.6}$$

Then

$$\underline{\dim}_{\mathsf{B}}(E) = \liminf_{t\to\infty} \Delta_E(t)$$

and

$$\overline{\dim}_{\mathsf{B}}(E) = \limsup_{t\to\infty} \Delta_E(t)\,,$$

and Theorem A therefore shows that for typical $f \in C_{\mathsf{u}}(X)$, the box counting function $\Delta_{\mathrm{graph}(f)}(t)$ of the graph of f diverges in the worst possible way as $t \to \infty$. In this paper we will prove that the behaviour of the box counting dimension function

$$t \to \Delta_{\mathrm{graph}(f)}(t) = \frac{\log N_{e^{-t}}(\mathrm{graph}(f))}{t}$$

of the graph of a typical function $f \in C_{\mathsf{u}}(X)$ is spectacularly more irregular than suggested by Theorem A. Namely, there are standard techniques, known as averaging methods, that (at least in some cases) can assign limiting values to divergent functions (the precise definitions will be given below), and the purpose of this paper is to show the following surprising result: not only is $\Delta_{\mathrm{graph}(f)}(t)$ divergent as $t \to \infty$, but the function $\Delta_{\mathrm{graph}(f)}(t)$ diverges so badly as $t \to \infty$, that even exceptionally powerful averaging methods, including, for example, higher order Hölder and Cesaro averages and higher order Riesz-Hardy logarithmic averages, are not able to "smoothen out" the irregularities in $\Delta_{\mathrm{graph}(f)}(t)$ as $t \to \infty$.

We start by recalling the definition of an averaging (or summability) method.

Definition. Average system. *An averaging system is a family* $\Pi = (\Pi_t)_{t \geq t_0}$ *with* $t_0 > 0$ *such that:*

(i) Π_t *is a finite Borel measure on* $[t_0, \infty)$*;*

(ii) Π_t *has compact support;*

(iii) *The Consistency Condition: If* $f : [t_0, \infty) \to [0, \infty)$ *is a positive measurable function and there is a real number* a *such that* $f(t) \to a$ *as* $t \to \infty$*, then* $\int f \, d\Pi_t \to a$ *as* $t \to \infty$*.*

If $f : [t_0, \infty) \to [0, \infty)$ *is a positive measurable function, then we define lower and upper* Π*-average of* f *by*

$$\underline{A}_\Pi f = \liminf_{t \to \infty} \int f \, d\Pi_t$$

and

$$\overline{A}_\Pi f = \limsup_{t \to \infty} \int f \, d\Pi_t \, ,$$

respectively.

The reader is referred to Hardy's excellent classical text [Ha] for a detailed and thorough discussion of average systems, and examples that demonstrate when averaging methods do assign limiting values to divergent functions.

We will now apply various averaging methods to the box counting function $\Delta_{\mathrm{graph}(f)}(t)$ of $f \in C_{\mathsf{u}}(X)$. Namely, for a bounded subset E of $\mathbb{R}^m$ and a positive averaging method $\Pi = (\Pi_t)_{t \geq t_0}$, we define the lower and upper Π-average box dimensions of E by

$$\begin{aligned}\underline{\dim}_{\Pi,\mathsf{B}}(E) &= \underline{A}_\Pi \Delta_E \\ &= \liminf_{t \to \infty} \int \frac{\log N_{e^{-s}}(E)}{s} \, d\Pi_t(s) \, ,\end{aligned}$$

and

$$\begin{aligned}\overline{\dim}_{\Pi,\mathsf{B}}(E) &= \overline{A}_\Pi \Delta_E \\ &= \limsup_{t \to \infty} \int \frac{\log N_{e^{-s}}(E)}{s} \, d\Pi_t(s) \, ,\end{aligned}$$

respectively. The next statement, i.e. Theorem 1.1, is the main result in the paper. This result shows that the behaviour of a typical (in the sense of Baire category) function $f \in C_{\mathsf{u}}(X)$ is so irregular that the box counting function $t \to \Delta_{\mathrm{graph}(f)}(t)$ of the graph of f remains divergent as $t \to \infty$ even after being "averaged" using very general and powerful averaging methods Π including, for example, higher order Hölder and Cesaro averages and higher order Riesz-Hardy logarithmic averages.

Theorem 1.1. *Let* $\Pi = (\Pi_t)_{t \geq t_0}$ *be an averaging system and let* X *be a bounded subset of* $\mathbb{R}^d$*.*

(1) *For all* $f \in C_{\mathsf{u}}(X)$*, we have*

$$\begin{aligned}\underline{\dim}_{\Pi,\mathsf{B}}(X) &\leq \underline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) \\ &\leq \overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) \\ &\leq \sup_{\varphi \in C_{\mathsf{u}}(X)} \overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(\varphi)) \leq \overline{\dim}_{\Pi,\mathsf{B}}(X) + 1 \, .\end{aligned}$$

(2) *For a typical function* $f \in C_{\mathsf{u}}(X)$*, we have*

$$\underline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) = \underline{\dim}_{\Pi,\mathsf{B}}(X) \, .$$

(3) (i) *For a typical function* $f \in C_{\mathsf{u}}(X)$*, we have*

$$\overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) = \sup_{\varphi \in C_{\mathsf{u}}(X)} \overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(\varphi)) \leq \overline{\dim}_{\Pi,\mathsf{B}}(X) + 1 \, .$$

(ii) *If, in addition,* X *only has finitely many isolated points, then for a typical function* $f \in C_{\mathsf{u}}(X)$*, we have*

$$\overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) = \overline{\dim}_{\Pi,\mathsf{B}}(X) + 1 \, .$$

The proof of Theorem 1.1 is given in Sections 5–8. Note that the statements in Theorem 1.1.(1) are trivial, and are merely included for completeness. Section 5 contains various preliminary results. The proof of Theorem 1.1.(2) is given in Section 6. The proof of Theorem 1.1.(3).(i) is given in Section 7, and the proof of Theorem 1.1.(3).(ii) is given in Section 8.

Remark. Note that if we let Π denote the average system defined by $\Pi = (\delta_t)_{t\geq 1}$ (where δ_t denotes the Dirac measure concentrated at t), then

$$\underline{\dim}_{\Pi,\mathsf{B}}(E) = \underline{\dim}_{\mathsf{B}}(E)\,,\quad \overline{\dim}_{\Pi,\mathsf{B}}(E) = \overline{\dim}_{\mathsf{B}}(E)\,,$$

for all subsets E of $\mathbb{R}^m$. Hence, if we apply Theorem 1.1 to the average system defined by $\Pi = (\delta_t)_{t\geq 1}$, then the statement in Theorem 1.1 simplifies to Theorem A.

If the box dimension of X exists and X only has finitely many isolated points, then the statement in Theorem 1.1 simplifies considerably; this is the content of the next corollary.

Corollary 1.2. *Let $\Pi = (\Pi_t)_{t\geq t_0}$ be an averaging system and let X be a bounded subset of $\mathbb{R}^d$. Assume that the box dimension of X exists and that X only has finitely many isolated points.*

(1) *For all $f \in C_{\mathsf{u}}(X)$, we have*

$$\dim_{\mathsf{B}}(X) \leq \underline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) \leq \overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) \leq \dim_{\mathsf{B}}(X) + 1\,.$$

(2) *For a typical function $f \in C_{\mathsf{u}}(X)$, we have*

$$\begin{aligned}\underline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) &= \dim_{\mathsf{B}}(X)\,,\\ \overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) &= \dim_{\mathsf{B}}(X) + 1\,.\end{aligned}$$

In Sections 2–3, we present several applications of Theorem 1.1 to different averaging methods Π, namely:

- In Section 2 we apply Theorem 1.1 to Hölder and Cesaro averages. This allows us to compute the higher order Hölder and Cesaro averages of the box counting function $\Delta_{\mathrm{graph}(f)}(t)$ of a typical function $f \in C_{\mathsf{u}}(X)$.
- In Section 3 we apply Theorem 1.1 to higher order Riesz-Hardy logarithmic averages. This allows us to compute the higher order Riesz-Hardy logarithmic averages of the box counting function $\Delta_{\mathrm{graph}(f)}(t)$ of a typical function $f \in C_{\mathsf{u}}(X)$.

2. Hölder and Cesaro averages of the box dimension of the graph of a typical function.

Two of the most commonly used averaging method are Hölder averages and Cesaro averages. We will now define these average methods and apply them to the box counting function $t \to \Delta_{\mathrm{graph}(f)}(t)$ of the graph of f. We first recall the definitions of the Hölder and Cesaro averages. For $a > 0$ and a measurable function $f : (a,\infty) \to [0,\infty)$, we define $Mf : (a,\infty) \to [0,\infty)$ by

$$(Mf)(t) = \frac{1}{t}\int_a^t f(s)\,ds\,.$$

For a positive integer n, we now define the lower and upper n'th order Hölder averages of f by

$$\begin{aligned}\underline{H}_n f &= \liminf_{t\to\infty}\,(M^n f)(t)\,,\\ \overline{H}_n f &= \limsup_{t\to\infty}(M^n f)(t)\,.\end{aligned}$$

The Cesaro averages are defined as follows. First, we define $If:(a,\infty)\to[0,\infty)$ by

$$(If)(t)=\int_a^t f(s)\,ds\,.$$

For a positive integer n, we now define the lower and upper n'th order Cesaro averages of f by

$$\underline{C}_n f=\liminf_{t\to\infty}\frac{n!}{t^n}(I^n f)(t)\,,$$
$$\overline{C}_n f=\limsup_{t\to\infty}\frac{n!}{t^n}(I^n f)(t)\,.$$

It is well-known that that the Hölder and Cesaro averages satisfy the following inequalities, namely,

$$\begin{aligned}\liminf_{t\to\infty} f(t)=\underline{H}_0 f\le\underline{H}_1 f\le\underline{H}_2 f\le\ldots\le\overline{H}_2 f\le\overline{H}_1 f\le\overline{H}_0 f=\limsup_{t\to\infty} f(t)\,,\\ \liminf_{t\to\infty} f(t)=\underline{C}_0 f\le\underline{C}_1 f\le\underline{C}_2 f\le\ldots\le\overline{C}_2 f\le\overline{C}_1 f\le\overline{C}_0 f=\limsup_{t\to\infty} f(t)\,,\end{aligned}\tag{2.1}$$

and

$$\underline{C}_n f\le\underline{H}_n f\le\overline{H}_n f\le\underline{C}_n f\,.\tag{2.2}$$

It is also well-known that the Hölder and Cesaro averages are averaging methods in the sense of the definition in Section 1.2. Indeed, if we for a positive integer n, define the averaging method $\Pi_n^{\mathsf{H}}=(\Pi_{n,t}^{\mathsf{H}})_{t\ge a}$ by

$$\Pi_{n,t}^{\mathsf{H}}(B)=\frac{1}{(n-1)!\,t}\int_{[a,t]\cap B}(\log t-\log s)^{n-1}\,ds$$

for Borel subsets B of $[a,\infty)$, then

$$\underline{H}_n f=\liminf_t\int f\,d\Pi_{n,t}^{\mathsf{H}}\,,$$
$$\overline{H}_n f=\limsup_t\int f\,d\Pi_{n,t}^{\mathsf{H}}\,,$$

see, for example, [Ja, p. 675]. Similarly, if we for a positive integer n, define the averaging method $\Pi_n^{\mathsf{C}}=(\Pi_{n,t}^{\mathsf{C}})_{t\ge a}$ by

$$\Pi_{n,t}^{\mathsf{C}}(B)=\frac{n}{t^n}\int_{[a,x]\cap B}(t-s)^{n-1}\,ds$$

then

$$\underline{C}_n f=\liminf_t\int f\,d\Pi_{n,t}^{\mathsf{C}}\,,$$
$$\overline{C}_n f=\limsup_t\int f\,d\Pi_{n,t}^{\mathsf{C}}\,,$$

see, for example, [Ha, pp. 110-111]. For example, this shows that the n'th order lower Hölder and Cesaro averages of f are given by

$$\underline{H}_n f=\liminf_{t\to\infty}\frac{1}{(n-1)!\,t}\int_a^t(\log t-\log s)^{n-1}\,f(s)\,ds$$

and

$$\underline{C}_n f=\liminf_{t\to\infty}\frac{n}{t^n}\int_a^t(t-s)^{n-1}\,f(s)\,ds\,.$$

There are similar formulas for the n'th order upper Hölder and Cesaro averages of f.

Using Hölder and Cesaro averages we can now introduce average Hölder and Cesaro box dimensions by applying the definitions of the Hölder and Cesaro averages to the function $t\to\Delta_{\mathrm{graph}(f)}(t)$. This is the content of the next definition.

Definition. Average Hölder and Cesaro box dimensions. *For a bounded subset E of $\mathbb{R}^m$, we define the lower and upper n'th order average Hölder box dimension of E, denoted by $\underline{\dim}^{\mathsf{H}}_{\mathsf{B},n}(E)$ and $\overline{\dim}^{\mathsf{H}}_{\mathsf{B},n}(E)$, as the lower and upper n'th order Hölder average of the function $t \to \Delta_E(t)$ for $t \geq 1$, i.e. we put*

$$\underline{\dim}^{\mathsf{H}}_{\mathsf{B},n}(E) = \underline{H}_n \Delta_E \,,$$
$$\overline{\dim}^{\mathsf{H}}_{\mathsf{B},n}(E) = \overline{H}_n \Delta_E \,.$$

Similarly, we define the lower and upper n'th order average Cesaro box dimension of E, denoted by $\underline{\dim}^{\mathsf{C}}_{\mathsf{B},n}(E)$ and $\overline{\dim}^{\mathsf{C}}_{\mathsf{B},n}(E)$, by

$$\underline{\dim}^{\mathsf{C}}_{\mathsf{B},n}(E) = \underline{C}_n \Delta_E \,,$$
$$\overline{\dim}^{\mathsf{C}}_{\mathsf{B},n}(E) = \overline{C}_n \Delta_E \,.$$

The higher order average Hölder and Cesaro box dimensions form a double infinite hierarchy in (at least) countably infnite many levels, namely, we have (using (2.1))

$$\begin{aligned} \underline{\dim}_{\mathsf{B}}(E) = \underline{\dim}^{\mathsf{H}}_{\mathsf{B},0}(E) \leq \underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(E) \leq \ldots \leq \overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(E) \leq \overline{\dim}^{\mathsf{H}}_{\mathsf{B},0}(E) = \overline{\dim}_{\mathsf{B}}(E) \,, \\ \underline{\dim}_{\mathsf{B}}(E) = \underline{\dim}^{\mathsf{C}}_{\mathsf{B},0}(E) \leq \underline{\dim}^{\mathsf{C}}_{\mathsf{B},1}(E) \leq \ldots \leq \overline{\dim}^{\mathsf{C}}_{\mathsf{B},1}(E) \leq \overline{\dim}^{\mathsf{C}}_{\mathsf{B},0}(E) = \overline{\dim}_{\mathsf{B}}(E) \,. \end{aligned} \tag{2.3}$$

As an application of Theorem 1.1, we will now show that the behaviour of a typical function $f \in C_{\mathsf{u}}(X)$ is so irregular that not even the hierarchies in (2.3) formed by taking Hölder and Cesaro averages of all orders are sufficiently powerful to "smoothen out" the behaviour of the box counting function $\Delta_{\mathrm{graph}(f)}(t)$ as $t \to \infty$.

Theorem 2.1. *Let X be a bounded subset of $\mathbb{R}^d$ with finitely many isolated points. Then a typical function $f \in C_{\mathsf{u}}(X)$ satisfies:*

$$\underline{\dim}^{\mathsf{H}}_{\mathsf{B},n}(\mathrm{graph}(f)) = \underline{\dim}^{\mathsf{H}}_{\mathsf{B},n}(X) \,,$$
$$\overline{\dim}^{\mathsf{H}}_{\mathsf{B},n}(\mathrm{graph}(f)) = \overline{\dim}^{\mathsf{H}}_{\mathsf{B},n}(X) + 1 \,,$$
$$\underline{\dim}^{\mathsf{C}}_{\mathsf{B},n}(\mathrm{graph}(f)) = \underline{\dim}^{\mathsf{C}}_{\mathsf{B},n}(X) \,,$$
$$\overline{\dim}^{\mathsf{C}}_{\mathsf{B},n}(\mathrm{graph}(f)) = \overline{\dim}^{\mathsf{C}}_{\mathsf{B},n}(X) + 1 \,,$$

for all $n \in \mathbb{N} \cup \{0\}$. In particular, if, in addition, the box dimension of X exists, then a typical function $f \in C_{\mathsf{u}}(X)$ satisfies:

$$\underline{\dim}^{\mathsf{H}}_{\mathsf{B},n}(\mathrm{graph}(f)) = \underline{\dim}^{\mathsf{C}}_{\mathsf{B},n}(\mathrm{graph}(f)) = \dim_{\mathsf{B}}(X) \,,$$
$$\overline{\dim}^{\mathsf{H}}_{\mathsf{B},n}(\mathrm{graph}(f)) = \overline{\dim}^{\mathsf{C}}_{\mathsf{B},n}(\mathrm{graph}(f)) = \dim_{\mathsf{B}}(X) + 1 \,,$$

for all $n \in \mathbb{N} \cup \{0\}$.

Proof.
This statement follows immediately from Theorem 1.1. $\square$

3. Riesz-Hardy averages of the box dimension of the graph of a typical function.

Higher order Riesz-Hardy logarithmic averages were introduced into the study of fractal properties of sets and measures by Fisher [Fi1] and Bedford & Fisher [BeFi] in the early 1990's (see also [ArDeFi]), and has since been investigated further by a large number of authors, including Graf [Gr], Mörters [Mö1,Mö2,Mö3] and Zähle [Zä]; the precise definition of the higher order Riesz-Hardy logarithmic averages will be given below. Motivated by this, we will now study the higher order Riesz-Hardy logarithmic averages of the box counting function $t \to \Delta_{\text{graph}(f)}(t)$ of the graph of f. We first recall the definition of higher order Riesz-Hardy logarithmic averages. Define $\log_+ : \mathbb{R} \to \mathbb{R}$ by $\log_+(t) = \log(t)$ for $t > 0$ and $\log_+(t) = 0$ for $t \leq 0$, and for a function $f : \mathbb{R} \to \mathbb{R}$, define the functions $Ef, Lf : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}(Ef)(t) &= f(e^t)\,,\\ (Lf)(t) &= f(\log_+(t))\,.\end{aligned}$$

Next, for a positive measurable function $f : \mathbb{R} \to [0, \infty)$, we define the function $\Lambda f : \mathbb{R} \to \mathbb{R}$ by

$$(\Lambda f)(u) = e^{-u} \int_{-\infty}^{u} e^t f(t)\, dt\,;$$

i.e. Λf is the convolution product between f and the function $\lambda : \mathbb{R} \to \mathbb{R}$ defined by $\lambda(t) = 0$ for $t \leq 0$ and $\lambda(t) = e^{-t}$ for $0 < t$. The higher order Riesz-Hardy logarithmic averages of a positive measurable function $f : \mathbb{R} \to [0, \infty)$ are now defined as follows. Namely, for a positive integer $n \in \mathbb{N}$, the lower and upper n'th order Riesz-Hardy logarithmic averages are defined by

$$\begin{aligned}\underline{R}_n f &= \liminf_{t\to\infty} (L^n \Lambda E^n f)(t)\,,\\ \overline{R}_n f &= \limsup_{t\to\infty} (L^n \Lambda E^n f)(t)\,.\end{aligned}$$

It is well-known that that the Riesz-Hardy logarithmic averages satisfy the following inequalities, namely,

$$\liminf_{t\to\infty} f(t) = \underline{R}_0 f \leq \underline{R}_1 f \leq \underline{R}_2 f \leq \ldots \leq \overline{R}_2 f \leq \overline{R}_1 f \leq \overline{R}_0 f = \limsup_{t\to\infty} f(t)\,; \tag{2.4}$$

the reader is referred to [BeFi, pp. 98–99, Property (1)] for a discussion of the proof of various special cases of (2.4), and note that [BeFi] refers the reader to [Fi2] for further discussions of the proof of (2.4) for an arbitrary positive measurable function f. It is also well-known that the Riesz-Hardy logarithmic averages are averaging methods in the sense of the definition in Section 1.2. Indeed, if we for a positive integer n, define the averaging method $\Pi_n^{\mathsf{R}} = (\Pi_{n,t}^{\mathsf{R}})_{t \geq a}$ for $a > 0$ by

$$\Pi_{n,t}^{\mathsf{R}}(B) = \frac{1}{\log_+^{n-1}(t)} \int_{[a,t]\cap B} \frac{\log_+^{n-1}(s)}{\displaystyle\prod_{k=0}^{n-1} \log_+^k(s)}\, ds$$

for Borel subsets B of $[a, \infty)$, then

$$\begin{aligned}\underline{R}_n f &= \liminf_t \int f\, d\Pi_{n,t}^{\mathsf{R}}\,,\\ \overline{R}_n f &= \limsup_t \int f\, d\Pi_{n,t}^{\mathsf{R}}\,,\end{aligned}$$

see, for example, [BeFi]. For example, this shows that the n'th order lower Riesz-Hardy logarithmic averages of f are given by

$$\underline{R}_n f = \liminf_{t\to\infty} \frac{1}{\log_+^{n-1}(t)} \int_a^t \frac{\log_+^{n-1}(s)}{\displaystyle\prod_{k=0}^{n-1} \log_+^k(s)} f(s)\, ds$$

There is a similar formula for the n'th order upper Riesz-Hardy logarithmic averages of f.

Using Riesz-Hardy averages we can now introduce average Riesz-Hardy box dimensions by applying the definitions of the Riesz-Hardy averages to the function $t \to \Delta_{\text{graph}(f)}(t)$. This is the content of the next definition.

Definition. Average Riesz-Hardy box dimension. *For a bounded subset E of $\mathbb{R}^m$, we define the lower and upper n'th order average Riesz-Hardy box dimension of E, denoted by $\underline{\dim}^{\mathsf{R}}_{\mathsf{B},n}(E)$ and $\overline{\dim}^{\mathsf{R}}_{\mathsf{B},n}(E)$, as the lower and upper n'th order Riesz-Hardy average of the function $t \to \Delta_E(t)$ for $t \geq 1$, i.e. we put*

$$\underline{\dim}^{\mathsf{R}}_{\mathsf{B},n}(E) = \underline{R}_n \Delta_E \,,$$
$$\overline{\dim}^{\mathsf{R}}_{\mathsf{B},n}(E) = \overline{R}_n \Delta_E \,.$$

The higher order average Riesz-Hardy box dimensions form a double infinite hierarchy in (at least) countably infnite many levels, namely, we have (using (2.4))

$$\underline{\dim}_{\mathsf{B}}(E) = \underline{\dim}^{\mathsf{R}}_{\mathsf{B},0}(E) \leq \underline{\dim}^{\mathsf{R}}_{\mathsf{B},1}(E) \leq \ldots \leq \overline{\dim}^{\mathsf{R}}_{\mathsf{B},1}(E) \leq \overline{\dim}^{\mathsf{R}}_{\mathsf{B},0}(E) = \overline{\dim}_{\mathsf{B}}(E) \,. \tag{2.5}$$

As a further application of Theorem 1.1, we will now show that the behaviour of a typical function $f \in C_{\mathsf{u}}(X)$ is so irregular that not even the hierarchy in (2.5) formed by taking higher order Riesz-Hardy averages is sufficiently powerful to "smoothen out" the behaviour the box counting of $\Delta_{\text{graph}(f)}(t)$ as $t \to \infty$.

Theorem 3.1. *Let X be a bounded subset of $\mathbb{R}^d$ with finitely many isolated points. Then a typical function $f \in C_{\mathsf{u}}(X)$ satisfies:*

$$\underline{\dim}^{\mathsf{R}}_{\mathsf{B},n}(\text{graph}(f)) = \underline{\dim}^{\mathsf{R}}_{\mathsf{B},n}(X) \,,$$
$$\overline{\dim}^{\mathsf{R}}_{\mathsf{B},n}(\text{graph}(f)) = \overline{\dim}^{\mathsf{R}}_{\mathsf{B},n}(X) + 1 \,,$$

for all $n \in \mathbb{N} \cup \{0\}$. In particular, if, in addition, the box dimension of X exists, then a typical function $f \in C_{\mathsf{u}}(X)$ satisfies:

$$\underline{\dim}^{\mathsf{R}}_{\mathsf{B},n}(\text{graph}(f)) = \dim_{\mathsf{B}}(X) \,,$$
$$\overline{\dim}^{\mathsf{R}}_{\mathsf{B},n}(\text{graph}(f)) = \dim_{\mathsf{B}}(X) + 1 \,,$$

for all $n \in \mathbb{N} \cup \{0\}$.

Proof.
This statement follows immediately from Theorem 1.1. □

4. An Example.

Of course, if X is a bounded subset of $\mathbb{R}^d$ with finitely many isolated points and such that the box dimension of X exists, then it follows from Theorem 1.1 that the lower Π-average box dimension of the graph of a typical function $f \in C_{\mathsf{u}}(X)$ equals the box dimension of X for *all* average systems Π, i.e.

$$\underline{\dim}_{\Pi,\mathsf{B}}(\text{graph}(f)) = \underline{\dim}_{\mathsf{B}}(\text{graph}(f)) = \dim_{\mathsf{B}}(X)$$

for *all* average systems Π, and the upper Π-average box dimension of the graph of a typical function $f \in C_{\mathsf{u}}(X)$ equals the box dimension of X plus 1 for *all* average systems Π, i.e.

$$\overline{\dim}_{\Pi,\mathsf{B}}(\text{graph}(f)) = \overline{\dim}_{\mathsf{B}}(\text{graph}(f)) = \dim_{\mathsf{B}}(X) + 1$$

for *all* average systems Π. However, we believe that the real novelty of Theorem 1.1 is that it also provides detailed information about the average box dimensions of the graph of a typical function $f \in C_{\mathsf{u}}(X)$ even when the box dimension of X fails to exist. Of course, in this case the Π-average box dimensions and the box dimensions of the graph of a typical function $f \in C_{\mathsf{u}}(X)$ may differ for some average system Π, i.e. it may happen that

$$\underline{\dim}_{\mathsf{B}}(\text{graph}(f)) < \underline{\dim}_{\Pi,\mathsf{B}}(\text{graph}(f))$$

or

$$\overline{\dim}_{\Pi,\mathsf{B}}(\operatorname{graph}(f)) < \overline{\dim}_{\mathsf{B}}(\operatorname{graph}(f)),$$

for some average system Π. It seems to us that this substantially more subtle scenario is the far most important and interesting case, and we believe that it is useful and illustrative to present a concrete example of this situation. Specifically, we will present an example of a (compact) subset X of $\mathbb{R}$ for which the box dimensions and the 1'st order average Hölder box dimensions of a graph typical function $f \in C_{\mathsf{u}}(X)$ *are all different*, i.e. we will give an example of a (compact) subset X of $\mathbb{R}$ without any isolated points such that

$$\underline{\dim}_{\mathsf{B}}(\operatorname{graph}(f)) < \underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(\operatorname{graph}(f)) < \overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(\operatorname{graph}(f)) < \overline{\dim}_{\mathsf{B}}(\operatorname{graph}(f))$$

for a typical function $f \in C_{\mathsf{u}}(X)$. Of course, in order to construct such an example, the set X must satisfy $\underline{\dim}_{\mathsf{B}}(X) < \underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X) < \overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X) < \overline{\dim}_{\mathsf{B}}(X)$, and this requirement is the reason behind the somewhat intricate construction of X. We now construct the set X. For $i = 0, 1, 2, 3, 4$, define the map $S_i : [0, 1] \to [0, 1]$ by $S_i(x) = \frac{1}{5}x + \frac{i}{5}$. Let $N_1, N_2, \dots \in \mathbb{N}$ be defined by $N_1 = 1$ and $N_n = 2^{n-2}$ for $n \geq 2$, and for a positive integer n, write

$$\Sigma_n = \begin{cases} \left\{ i_1 \dots i_{N_n} \,\middle|\, i_j \in \{0, 4\} \text{ for all } j \right\} & \text{if } n \text{ is even;} \\ \left\{ i_1 \dots i_{N_n} \,\middle|\, i_j \in \{0, 2, 4\} \text{ for all } j \right\} & \text{if } n \text{ is odd,} \end{cases}$$

i.e. Σ_n is the family of all finite strings $\mathbf{i} = i_1 \dots i_{N_n}$ of length N_n with entries i_j from $\{0, 2, 4\}$ if n is odd, and with entries i_j from $\{0, 4\}$ if n is even. For $\mathbf{i} = i_1 \dots i_{N_n} \in \Sigma_n$, we write $S_{\mathbf{i}} = S_{i_1} \circ \dots \circ S_{i_{N_n}}$. The set X is now defined as follows. For a positive integer n, let

$$X_n = \bigcup_{\mathbf{i}_i \in \Sigma_1, \dots, \mathbf{i}_n \in \Sigma_n} S_{\mathbf{i}_1} \circ \dots \circ S_{\mathbf{i}_n}([0, 1]),$$

and put

$$X = \bigcap_n X_n. \tag{4.1}$$

Finally, for brevity we write

$$a = \tfrac{\log 2}{\log 5}, \quad b = \tfrac{\log 3}{\log 5}.$$

The box dimensions and the 1'st order Hölder average box dimensions of the graph of a typical function in $C_{\mathsf{u}}(X)$ are given by Theorem 4.1 below.

Theorem 4.1. *Let X be given by (4.1). Then a typical function $f \in C_{\mathsf{u}}(X)$ satisfies*

$$\begin{aligned}
\underline{\dim}_{\mathsf{B}}(\operatorname{graph}(f)) &= \underline{\dim}_{\mathsf{B}}(X) &&= \tfrac{2}{3}a + \tfrac{1}{3}b &&\approx 0.51465, \\
\underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(\operatorname{graph}(f)) &= \underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X) &&= \tfrac{2^{\frac{2}{3}}}{3}a + \left(1 - \tfrac{2^{\frac{2}{3}}}{3}\right)b &&\approx 0.54930, \\
\overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(\operatorname{graph}(f)) &= 1 + \overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X) &&= 1 + \left(1 - \tfrac{2^{\frac{2}{3}}}{3}\right)a + \tfrac{2^{\frac{2}{3}}}{3}b &&\approx 0.56398, \\
\overline{\dim}_{\mathsf{B}}(\operatorname{graph}(f)) &= 1 + \overline{\dim}_{\mathsf{B}}(X) &&= 1 + \tfrac{1}{3}a + \tfrac{2}{3}b &&\approx 0.59863.
\end{aligned}$$

In particular, a typical function $f \in C_{\mathsf{u}}(X)$ satisfies

$$\underline{\dim}_{\mathsf{B}}(\operatorname{graph}(f)) < \underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(\operatorname{graph}(f)) < \overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(\operatorname{graph}(f)) < \overline{\dim}_{\mathsf{B}}(\operatorname{graph}(f)).$$

The proof of Theorem 4.1 is given at the end of this section. We note that Theorem 4.1 shows that the box counting function $t \to \Delta_{\operatorname{graph}(f)}(t)$ of the graph of a typical function $f \in C_{\mathsf{u}}(X)$ is so irregular that even the 1'st order Hölder average of $\Delta_{\mathrm{graph}(f)}(t)$ fails to exists. Before proving Theorem 4.1, we present some numerical calculations illustrating this remarkable oscillatory behaviour of $\Delta_{\mathrm{graph}(f)}(t)$. We first introduce the following notation. For $E\subseteq\mathbb{R}^m$ and $\delta>0$, write

$$\Pi_\delta(E)=\left|\left\{Q\in\mathcal{Q}_\delta^m\,\middle|\,\mathring{Q}\cap E\neq\varnothing\right\}\right|,$$

where $\mathring{Q}$ denotes the interior of Q. The reason for introducing the numbers $\Pi_\delta(E)$ is twofold, namely: (1) while the box-dimensions of a general subset E of $\mathbb{R}^m$ cannot be computed using the numbers $\Pi_\delta(E)$ (for example, if $m=2$ and $E=\mathbb{R}\times\{0\}$, then $\underline{\dim}_{\mathsf{B}}(E)=\overline{\dim}_{\mathsf{B}}(E)=1$, but $\Pi_\delta(E)=0$ for all $\delta>0$), it is, nevertheless, true that the box dimensions of X, and hence the box dimensions of a typical function $f\in C_{\mathsf{u}}(X)$, can be expressed in terms of the numbers $\Pi_\delta(X)$, (see (4.3) and (4.6) below), and (2) there are simple explicit formulas for $\Pi_\delta(X)$ for $\delta=5^{-n}$ (see (4.10) below) allowing us to obtain explicit expressions for the box dimensions of X, and hence explicit expressions for the box dimensions of a typical function $f\in C_{\mathsf{u}}(X)$ (on the other hand, we have been unable to obtain similarly simple explicit formulas for $N_\delta(X)$ for $\delta>0$). We start by showing that the box dimensions of X can be expressed in terms of the numbers $\Pi_\delta(X)$. For brevity, we write $r_n=5^{-n}$ and put

$$\begin{aligned}\nu_n&=\frac{\log N_{r_n}(X)}{-\log r_n}\,.\\ \pi_n&=\frac{\log \Pi_{r_n}(X)}{-\log r_n}\,.\end{aligned}\tag{4.2}$$

Lemma 4.2. *If X is a bounded subset of $\mathbb{R}$ without any isolated points, then we have $\underline{\dim}_{\mathsf{B}}(X)=\liminf_n\frac{\log\Pi_{r_n}(X)}{-\log r_n}$ and $\overline{\dim}_{\mathsf{B}}(X)=\limsup_n\frac{\log\Pi_{r_n}(X)}{-\log r_n}$. In particular, if X denotes the set in (4.1), then*

$$\begin{aligned}\underline{\dim}_{\mathsf{B}}(X)&=\liminf_n\pi_n\,,\\ \overline{\dim}_{\mathsf{B}}(X)&=\limsup_n\pi_n\,.\end{aligned}\tag{4.3}$$

Proof.
It is trivially clear that $\Pi_\delta(X)\le N_\delta(X)$ for all $\delta>0$, whence $\liminf_n\frac{\log\Pi_{r_n}(X)}{-\log r_n}\le\underline{\dim}_{\mathsf{B}}(X)$ and $\limsup_n\frac{\log\Pi_{r_n}(X)}{-\log r_n}\le\overline{\dim}_{\mathsf{B}}(X)$. Next, we prove the reverse inequalities. For $Q\in\mathcal{Q}_\delta^1$ with $Q=[a_Q,b_Q]$ for $a_Q,b_Q\in\mathbb{R}$, write $Q^-=[a_Q-\delta,b_Q-\delta]$ and $Q^+=[a_Q+\delta,b_Q+\delta]$, i.e. Q^- and Q^+ are the δ-grid cubes in $\mathbb{R}$ immediately to the left and to the right of Q, respectively. Since X does not have any isolated points, it is easily seen that if $Q\in\mathcal{Q}_\delta^1$ with $Q\cap X\neq\varnothing$, then there is $P\in\{Q^-,Q,Q^+\}$ such that $\mathring{P}\cap X\neq\varnothing$, and so $N_\delta(X)\le 3\Pi_\delta(X)$. This clearly implies that $\underline{\dim}_{\mathsf{B}}(X)\le\liminf_n\frac{\log\Pi_{r_n}(X)}{-\log r_n}$ and $\overline{\dim}_{\mathsf{B}}(X)\le\limsup_n\frac{\log\Pi_{r_n}(X)}{-\log r_n}$. □

Next, for $t>0$, let n_t be the unique integer such that $r_{n_t+1}\le e^{-t}<r_{n_t}$ and note that a straight forward albeit somewhat lengthy calculation shows that (for the details of this argument, the reader may consult Lemma 5.6 where a more general result is proved)

$$\begin{aligned}\underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X)&=\liminf_t\frac{1}{t}\int_1^t\frac{\log N_{r_{n_s}}(X)}{-\log r_{n_s}}\,ds=\liminf_t\frac{1}{t}\int_1^t\nu_{n_s}\,ds\,,\\ \overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X)&=\limsup_t\frac{1}{t}\int_1^t\frac{\log N_{r_{n_s}}(X)}{-\log r_{n_s}}\,ds=\limsup_t\frac{1}{t}\int_1^t\nu_{n_s}\,ds\,.\end{aligned}\tag{4.4}$$

Also, it is not difficult to see that

$$\begin{aligned}\liminf_t\frac{1}{t}\int_1^t\nu_{n_s}\,ds&=\liminf_n\frac{1}{n}\sum_{i=1}^n\nu_i=\liminf_n\frac{1}{n}\sum_{i=1}^n\pi_i\,,\\ \limsup_t\frac{1}{t}\int_1^t\nu_{n_s}\,ds&=\limsup_n\frac{1}{n}\sum_{i=1}^n\nu_i=\limsup_n\frac{1}{n}\sum_{i=1}^n\pi_i\,.\end{aligned}\tag{4.5}$$

Combining (4.4) and (4.5) now shows that the average box dimensions $\underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X)$ and $\overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X)$ are given by

$$\begin{aligned} \underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X) &= \liminf_n \frac{1}{n}\sum_{i=1}^{n}\pi_i\,, \\ \overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X) &= \limsup_n \frac{1}{n}\sum_{i=1}^{n}\pi_i\,. \end{aligned} \tag{4.6}$$

It follows from (4.3) and (4.6) that the box dimensions of X, namely $\underline{\dim}_{\mathsf{B}}(X)$ and $\overline{\dim}_{\mathsf{B}}(X)$, and the average box dimension of X, namely $\underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X)$ and $\overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X)$, equal the lower and upper limits of the sequences $(\pi_n)_n$ and $(\frac{1}{n}\sum_{i=1}^{n}\pi_i)_n$, respectively. Below we sketch the graphs of the sequences $(\pi_n)_n$ and $(\frac{1}{n}\sum_{i=1}^{n}\pi_i)_n$ illustrating their oscillatory behaviour.

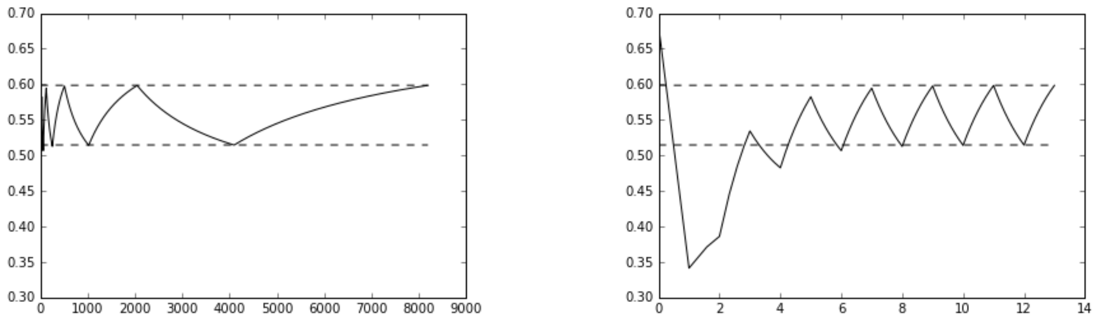

Figure 4.1. The figure on the left shows the points (n, π_n) for $n \in \{1, 2, 3, \ldots, 2^{13}\}$, and the figure on the right shows the points $(\frac{\log n}{\log 2}, \pi_n)$ for $n \in \{1, 2, 3, \ldots, 2^{13}\}$; the numbers π_n are computed using formula (4.10). The two horizontal dashed lines intersect the vertical axis at $\liminf_n \pi_n = \frac{2}{3}a + \frac{1}{3}b \approx 0.51465$ and $\limsup_n \pi_n = \frac{1}{3}a + \frac{2}{3}b \approx 0.59863$, respectively.

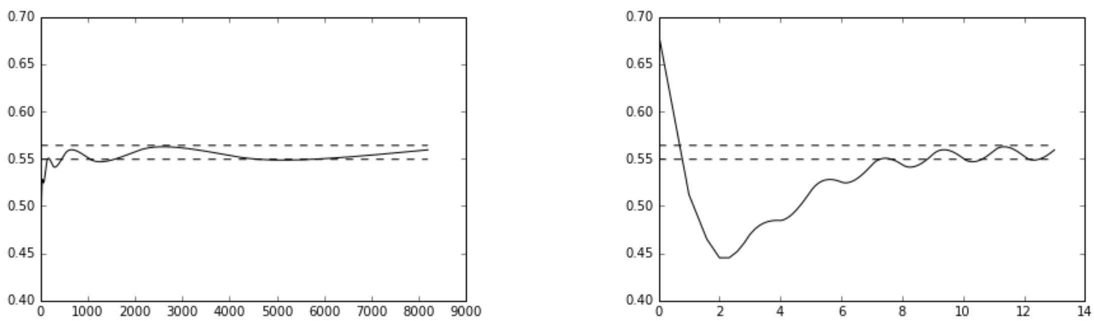

Figure 4.2. The figure on the left shows the points $(n, \frac{1}{n}\sum_{i=1}^{n}\pi_i)$ for $n \in \{1, 2, 3, \ldots, 2^{13}\}$, and the figure on the right shows the points $(\frac{\log n}{\log 2}, \frac{1}{n}\sum_{i=1}^{n}\pi_i)$ for $n \in \{1, 2, 3, \ldots, 2^{13}\}$; the numbers π_n are computed using formula (4.10). The two horizontal dashed lines intersect the vertical axis at $\liminf_n \frac{1}{n}\sum_{i=1}^{n}\pi_i = \frac{2^{\frac{2}{3}}}{3}a + (1 - \frac{2^{\frac{2}{3}}}{3})b \approx 0.54930$ and $\limsup_n \frac{1}{n}\sum_{i=1}^{n}\pi_i = (1 - \frac{2^{\frac{2}{3}}}{3})a + \frac{2^{\frac{2}{3}}}{3}b \approx 0.56398$, respectively.

We will now prove Theorem 4.1.

Proof of Theorem 4.1.
To prove Theorem 4.1, it clearly suffices to show that the box dimensions $\underline{\dim}_{\mathsf{B}}(X)$, $\overline{\dim}_{\mathsf{B}}(X)$, $\underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X)$ and $\overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X)$ are given by the formulas in the statement of Theorem 4.1. Indeed, If we can show that the box dimensions $\underline{\dim}_{\mathsf{B}}(X)$, $\overline{\dim}_{\mathsf{B}}(X)$, $\underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X)$ and $\overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X)$ are given by the formulas in the statement of Theorem 4.1, then the remaining statements in Theorem 4.1 follow immediately from Theorem 2.1. The formulas for the lower and upper box dimensions of X follow from routine arguments and the proofs are therefore omitted. We will now prove the formulas for the 1'st order average box dimensions of X, i.e. we will prove that

$$\begin{aligned}\underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X) &= \tfrac{2^{\frac{2}{3}}}{3}a+\left(1-\tfrac{2^{\frac{2}{3}}}{3}\right)b\,,\\ \overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X) &= \left(1-\tfrac{2^{\frac{2}{3}}}{3}\right)a+\tfrac{2^{\frac{2}{3}}}{3}b\,.\end{aligned}$$

Recall, that it follows from (4.6) that

$$\begin{aligned}\underline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X) &= \liminf_K \frac{1}{K}\sum_{k=1}^K \pi_k\,,\\ \overline{\dim}^{\mathsf{H}}_{\mathsf{B},1}(X) &= \limsup_K \frac{1}{K}\sum_{k=1}^K \pi_k\,.\end{aligned}\tag{4.7}$$

Below we will compute the numbers $\liminf_K \frac{1}{K}\sum_{k=1}^K \pi_k$ and $\limsup_K \frac{1}{K}\sum_{k=1}^K \pi_k$.

We start by introducing the following notation. Let

$$M_n=\sum_{k\le n}N_k=2^{n-1}\,,\quad M_n^{\mathsf{o}}=\sum_{\substack{k\le n\\ k\text{ is odd}}}N_k\,,\quad M_n^{\mathsf{e}}=\sum_{\substack{k\le n\\ k\text{ is even}}}N_k\,,$$

We first note that straight forward calculations show that there are bounded sequences $(\rho_n^{\mathsf{e}})_n$ and $(\rho_n^{\mathsf{o}})_n$ such that

$$M_n^{\mathsf{e}}=\begin{cases}\frac{1}{3}2^n+\rho_n^{\mathsf{e}} & \text{if } n \text{ is even;}\\ \frac{1}{6}2^n+\rho_n^{\mathsf{o}} & \text{if } n \text{ is odd,}\end{cases}\qquad M_n^{\mathsf{o}}=\begin{cases}\frac{1}{6}2^n+\rho_n^{\mathsf{e}} & \text{if } n \text{ is even;}\\ \frac{1}{3}2^n+\rho_n^{\mathsf{o}} & \text{if } n \text{ is odd.}\end{cases}\tag{4.8}$$

Similarly, straight forward calculations show that there are sequences $(\sigma_n^{\mathsf{e}})_n$ and $(\sigma_n^{\mathsf{o}})_n$ with $\frac{\sigma_n^{\mathsf{e}}}{2^n}\to 0$ and $\frac{\sigma_n^{\mathsf{o}}}{2^n}\to 0$ such that

$$\sum_{\substack{i<n\\ i\text{ is even}}}M_n^{\mathsf{e}}=\begin{cases}\frac{1}{9}2^n+\sigma_n^{\mathsf{e}} & \text{if } n \text{ is even;}\\ \frac{2}{9}2^n+\sigma_n^{\mathsf{o}} & \text{if } n \text{ is odd,}\end{cases}\qquad \sum_{\substack{i<n\\ i\text{ is odd}}}M_n^{\mathsf{o}}=\begin{cases}\frac{2}{9}2^n+\sigma_n^{\mathsf{e}} & \text{if } n \text{ is even;}\\ \frac{1}{9}2^n+\sigma_n^{\mathsf{o}} & \text{if } n \text{ is odd.}\end{cases}\tag{4.9}$$

Next, write

$$\lambda_n=\begin{cases}b & \text{if } n \text{ is even;}\\ a & \text{if } n \text{ is odd;}\end{cases}$$

recall, that $a=\frac{\log 2}{\log 5}$ and $b=\frac{\log 3}{\log 5}$. Finally, we note that a straightforward calculation shows that:

$$\text{if } M_n\le k\le M_{n+1}\,,\text{ then}\quad \pi_k=\frac{M_n^{\mathsf{e}}(a-\lambda_n)+M_n^{\mathsf{o}}(b-\lambda_n)}{k}+\lambda_n\tag{4.10}$$

We can now compute the numbers $\liminf_K \frac{1}{K}\sum_{k=1}^K \pi_k$ and $\limsup_K \frac{1}{K}\sum_{k=1}^K \pi_k$. We begin by deriving an explicit expression for the sum $\frac{1}{K}\sum_{k=1}^K \pi_k$. Let K be a positive integer and let $n(K)$ be the unique integer such that

$$M_{n(K)}\le K<M_{n(K)+1}\,.$$

We now have

$$\frac{1}{K}\sum_{k=1}^{K}\pi_k = A_K + B_K \tag{4.11}$$

where

$$A_K = \frac{1}{K}\sum_{k=1}^{M_{n(K)}-1}\pi_k\,,\quad B_K = \frac{1}{K}\sum_{k=M_{n(K)}}^{K}\pi_k\,.$$

We will now analyse the sums A_K and B_K; this is the contents of Claim 1 and Claim 2, respectively.

Claim 1. There is a sequence $(s_K)_K$ with $s_K \to 0$ such that

$$A_K = \begin{cases} (\frac{1}{3}b + \frac{2\log 2}{9}(b-a) + \frac{2}{3}a)\frac{2^{n(K)-1}}{K} + s_K & \textit{if } n(K) \textit{ is even;}\\ (\frac{2}{3}b - \frac{2\log 2}{9}(b-a) + \frac{1}{3}a)\frac{2^{n(K)-1}}{K} + s_K & \textit{if } n(K) \textit{ is odd.}\end{cases}$$

Proof of Claim 1. Write $u_i = \sum_{k=M_i}^{M_{i+1}-1}\frac{1}{k}$ and note that $u_i = \int_{M_i}^{M_{i+1}}\frac{1}{x}\,dx + \varepsilon_i = \log(\frac{M_{i+1}}{M_i}) + \varepsilon_i = \log 2 + \varepsilon_i$ where $\varepsilon_i \to 0$. Hence, using (4.10), we have

$$\begin{aligned}
A_K &= \frac{1}{K}\sum_{i<n(K)}(M_i^{\mathsf{e}}(a-\lambda_i) + M_i^{\mathsf{o}}(b-\lambda_i))u_i + \frac{1}{K}\sum_{i<n(K)}\lambda_i N_{i+1}\\
&= \frac{1}{K}\sum_{\substack{i<n(K)\\ i \text{ is odd}}}(M_i^{\mathsf{e}}(a-\lambda_i) + M_i^{\mathsf{o}}(b-\lambda_i))u_i + a\frac{1}{K}\sum_{\substack{i<n(K)\\ i \text{ is odd}}}N_{i+1}\\
&\quad + \frac{1}{K}\sum_{\substack{i<n(K)\\ i \text{ is even}}}(M_i^{\mathsf{e}}(a-\lambda_i) + M_i^{\mathsf{o}}(b-\lambda_i))u_i + b\frac{1}{K}\sum_{\substack{i<n(K)\\ i \text{ is even}}}N_{i+1}\\
&= b\frac{1}{K}M_{n(K)}^{\mathsf{o}} + (b-a)\frac{1}{K}\left(\sum_{\substack{i<n(K)\\ i \text{ is odd}}}M_i^{\mathsf{o}}u_i - \sum_{\substack{i<n(K)\\ i \text{ is even}}}M_i^{\mathsf{e}}u_i\right) + a\frac{1}{K}M_{n(K)}^{\mathsf{e}}\\
&= b\frac{1}{K}M_{n(K)}^{\mathsf{o}} + (b-a)\frac{1}{K}\left(\sum_{\substack{i<n(K)\\ i \text{ is odd}}}M_i^{\mathsf{o}}(\log 2+\varepsilon_i) - \sum_{\substack{i<n(K)\\ i \text{ is even}}}M_i^{\mathsf{e}}(\log 2+\varepsilon_i)\right) + a\frac{1}{K}M_{n(K)}^{\mathsf{e}}\\
&= b\frac{1}{K}M_{n(K)}^{\mathsf{o}} + (b-a)(\log 2)\frac{1}{K}\left(\sum_{\substack{i<n(K)\\ i \text{ is odd}}}M_i^{\mathsf{o}} - \sum_{\substack{i<n(K)\\ i \text{ is even}}}M_i^{\mathsf{e}}\right) + a\frac{1}{K}M_{n(K)}^{\mathsf{e}} + \sigma_K\,,
\end{aligned} \tag{4.12}$$

where $\sigma_K = (b-a)\frac{1}{K}(\sum_{\substack{i<n(K)\\ i \text{ is odd}}}M_i^{\mathsf{o}}\varepsilon_i - \sum_{\substack{i<n(K)\\ i \text{ is even}}}M_i^{\mathsf{e}}\varepsilon_i)$. Since it follows from (4.9) that $\sup_K \frac{1}{K}\sum_{\substack{i<n(K)\\ i \text{ is odd}}}M_i^{\mathsf{o}} < \infty$ and $\sup_K \frac{1}{K}\sum_{\substack{i<n(K)\\ i \text{ is even}}}M_i^{\mathsf{e}} < \infty$, we now conclude that

$$\sigma_K \to 0\,. \tag{4.13}$$

Finally, the desired result follows from a lengthy but straight forward calculation using (4.8), (4.9), (4.12) and (4.13). This completes the proof of Claim 1.

Claim 2. There is a sequence $(t_K)_K$ with $t_K \to 0$ such that

$$B_K = \begin{cases} \frac{2}{3}(b-a)\frac{2^{n(K)-1}}{K}\log\frac{2^{n(K)-1}}{K} + b - b\frac{2^{n(K)-1}}{K} + t_K & \textit{if } n(K) \textit{ is even;}\\ -\frac{2}{3}(b-a)\frac{2^{n(K)-1}}{K}\log\frac{2^{n(K)-1}}{K} + a - a\frac{2^{n(K)-1}}{K} + t_K & \textit{if } n(K) \textit{ is odd.}\end{cases}$$

Proof of Claim 2. Put $v_K = \sum_{k=M_{n(K)}}^{K} \frac{1}{k}$ and note that $v_K = \int_{M_{n(K)}}^{K} \frac{1}{x}\,dx + \delta_K = \log(\frac{K}{M_{n(K)}}) + \delta_K$ where $\delta_K \to 0$. A simple calculation using (4.8) shows that

$$\begin{aligned} B_K &= \frac{1}{K}(M^{\mathsf{e}}_{n(K)}(a-\lambda_{n(K)}) + M^{\mathsf{o}}_{n(K)}(b-\lambda_{n(K)}))v_K + \lambda_{n(K)}\frac{1}{K}(K - M_{n(K)}) \\ &= (M^{\mathsf{e}}_{n(K)}(a-\lambda_{n(K)}) + M^{\mathsf{o}}_{n(K)}(b-\lambda_{n(K)}))\frac{1}{K}\left(\log\left(\frac{K}{M_{n(K)}}\right) + \delta_K\right) \\ &\qquad\qquad + \lambda_{n(K)}\left(1 - \frac{1}{K}M_{n(K)}\right) \\ &= (M^{\mathsf{e}}_{n(K)}(a-\lambda_{n(K)}) + M^{\mathsf{o}}_{n(K)}(b-\lambda_{n(K)}))\frac{1}{K}\log\left(\frac{K}{M_{n(K)}}\right) \\ &\qquad\qquad + \lambda_{n(K)}\left(1 - \frac{1}{K}M_{n(K)}\right) + \tau_K \end{aligned} \tag{4.14}$$

where $\tau_K = (M^{\mathsf{e}}_{n(K)}(a-\lambda_{n(K)}) + M^{\mathsf{o}}_{n(K)}(b-\lambda_{n(K)}))\frac{1}{K}\delta_K$. Since it follows from (4.8) that $\sup_K \frac{1}{K}M^{\mathsf{e}}_{n(K)} < \infty$ and $\sup_K \frac{1}{K}M^{\mathsf{o}}_{n(K)} < \infty$, we now conclude that

$$\tau_K \to 0\,. \tag{4.15}$$

Finally, the desired result follows from a lengthy but straight forward calculation using (4.8), (4.14) and (4.15). This completes the proof of Claim 2.

We will now combine the expressions for A_K and B_K in Claim 1 and Claim 2, respectively, to derive an explicit expression for $\frac{1}{K}\sum_{k=1}^{K}\pi_k = A_K + B_K$. Define $f:(0,\infty)\to\mathbb{R}$ by

$$f(x) = -\tfrac{2(3-\log 2)}{9}(b-a)x + \tfrac{2}{3}(b-a)\,x\log x\,.$$

Combining Claim 1 and Claim 2 now shows that

$$\frac{1}{K}\sum_{k=1}^{K}\pi_k = A_K + B_K = \begin{cases} f(\frac{2^{n(K)-1}}{K}) + b + s_K + t_K & \text{if } n(K) \text{ is even;} \\ -f(\frac{2^{n(K)-1}}{K}) + a + s_K + t_K & \text{if } n(K) \text{ is odd.} \end{cases} \tag{4.16}$$

It follows easily from (4.16) that

$$\begin{aligned} \liminf_K \frac{1}{K}\sum_{k=1}^{K}\pi_k &= \min\left(\inf_{\frac{1}{2}\le x\le 1} f(x) + b\,,\, -\sup_{\frac{1}{2}\le x\le 1} f(x) + a\right), \\ \limsup_K \frac{1}{K}\sum_{k=1}^{K}\pi_k &= \max\left(\sup_{\frac{1}{2}\le x\le 1} f(x) + b\,,\, -\inf_{\frac{1}{2}\le x\le 1} f(x) + a\right). \end{aligned} \tag{4.17}$$

Finally, a routine calculus argument shows that

$$\begin{aligned} \min\left(\inf_{\frac{1}{2}\le x\le 1} f(x) + b\,,\, -\sup_{\frac{1}{2}\le x\le 1} f(x) + a\right) &= \frac{2^{\frac{2}{3}}}{3}a + \left(1 - \frac{2^{\frac{2}{3}}}{3}\right)b\,, \\ \max\left(\sup_{\frac{1}{2}\le x\le 1} f(x) + b\,,\, -\inf_{\frac{1}{2}\le x\le 1} f(x) + a\right) &= \left(1 - \frac{2^{\frac{2}{3}}}{3}\right)a + \frac{2^{\frac{2}{3}}}{3}b\,. \end{aligned} \tag{4.18}$$

The desired result now follows from combining (4.7), (4.17) and (4.18). □

5. Proof of Theorem 1.1: Preliminary results.

In this section we prove five technical auxiliary lemmas that will be used extensively in Sections 6–8. Recall, that if E is a subset of $\mathbb{R}^m$ and $\delta > 0$, then $N_\delta(E)$ is the number of δ-mesh cubes that intersect E, see (1.2) and (1.3). Also, $\overline{E}$ denotes the closure of E in $\mathbb{R}^m$.

Lemma 5.1. *Fix a bounded subset E of $\mathbb{R}^m$. Let $c > 1$.*

1. *For all $\delta > 0$, we have $N_\delta(E) \leq (c+2)^m\, N_{c\delta}(E)$.*
2. *For all $\delta > 0$, we have $N_{c\delta}(E) \leq 2^m\, N_\delta(E)$.*

Proof.
This follows from standard arguments, and for the sake of brevity we have therefore decided to omit the proof. ☐

Lemma 5.2. *Let $\Pi = (\Pi_t)_{t \geq t_0}$ be an averaging system. Fix a bounded subset E of $\mathbb{R}^m$. Then we have*

$$\underline{\dim}_{\Pi,\mathsf{B}}(E) = \underline{\dim}_{\Pi,\mathsf{B}}(\overline{E})\,,$$
$$\overline{\dim}_{\Pi,\mathsf{B}}(E) = \overline{\dim}_{\Pi,\mathsf{B}}(\overline{E})\,.$$

Proof.
It is easily seen that $N_\delta(E) \leq N_\delta(\overline{E}) \leq 3^m N_\delta(E)$ for all $\delta > 0$, whence

$$\int \frac{\log N_{e^{-s}}(E)}{s}\, d\Pi_t(s) \leq \int \frac{\log N_{e^{-s}}(\overline{E})}{s}\, d\Pi_t(s) \leq k \int f\, d\Pi_t + \int \frac{\log N_{e^{-s}}(E)}{s}\, d\Pi_t(s)\,, \tag{5.1}$$

where $k = \log(3^m)$ and where the function $f : (0,\infty) \to [0,\infty)$ is defined by $f(t) = \frac{1}{t}$. Since $f(t) \to 0$ as $t \to \infty$, we conclude from the Consistency Condition (i.e. Condition (iii)) in the definition of an average system that $\int f\, d\Pi_t \to 0$ as $t \to \infty$, and the desired result now follows from this and (5.1). ☐

Lemma 5.3. *Fix a bounded subset X of $\mathbb{R}^d$. Let $f \in C_{\mathsf{u}}(X)$ and $r > 0$. Then there is a polynomial $p : \mathbb{R}^d \to \mathbb{R}$ such that $\|f - p|_X\|_\infty < r$.*

Proof.
Since f is uniformly continuous on X, it follows from [Si, p. 78] that there is a continuous function $F : \overline{X} \to \mathbb{R}$ such that $F|_X = f$. Next, since $\overline{X}$ is compact, we conclude from Stone-Weierstrass' Theorem that there is is a polynomial $p : \mathbb{R}^d \to \mathbb{R}$ such that $\|F - p|_{\overline{X}}\|_\infty < r$, see, for example, [Ca, p. 198, Exercise 24]. In particular, we now conclude that $\|f - p|_X\|_\infty = \|F|_X - p|_X\|_\infty \leq \|F - p|_{\overline{X}}\|_\infty < r$.
☐

Lemma 5.4. *Fix a bounded subset X of $\mathbb{R}^d$. Let $f \in C_{\mathsf{u}}(X)$ and let $p : \mathbb{R}^d \to \mathbb{R}$ be a polynomial. Let $\lambda \in \mathbb{R}$ with $\lambda \neq 0$. Then there are constants $c > 1$ and $C > 1$ such that for all $\delta > 0$, we have*

$$\frac{1}{C}\, N_{c\delta}\big(\operatorname{graph}(p|_X + \lambda f)\big) \leq N_\delta\big(\operatorname{graph}(f)\big) \leq C\, N_{\frac{\delta}{c}}\big(\operatorname{graph}(p|_X + \lambda f)\big)\,.$$

Proof.
Define $F : \operatorname{graph}(f) \to \operatorname{graph}(p|_X + \lambda f)$ by $F(x, f(x)) = (x, p(x) + \lambda f(x))$ and note that F is bijective with $F^{-1}(x, p(x) + \lambda f(x)) = (x, f(x))$. An easy calculation shows that both F and F^{-1} are Lipschitz maps, and it is not difficult to see that this implies that are constants $c, C > 1$ such that for all $\delta > 0$, we have

$$\frac{1}{C}\, N_{c\delta}\big(F(\operatorname{graph}(f))\big) \leq N_\delta\big(\operatorname{graph}(f)\big) \leq C\, N_{\frac{\delta}{c}}\big(F(\operatorname{graph}(f))\big)\,.$$

Since clearly $F(\operatorname{graph}(f)) = \operatorname{graph}(p|_X + \lambda f)$, the desired conclusion follows from the above inequalities. ☐

Lemma 5.5. *Let $\Pi = (\Pi_t)_{t \geq t_0}$ be an averaging system. Fix a bounded subset X of $\mathbb{R}^d$. Let $f \in C_{\mathsf{u}}(X)$ and let $p : \mathbb{R}^d \to \mathbb{R}$ be a polynomial. Let $\lambda \in \mathbb{R}$ with $\lambda \neq 0$.*

(1) *We have*

$$\begin{aligned}\underline{\dim}_{\Pi,\mathsf{B}}\Big(\operatorname{graph}\big(p|_X + \lambda f\big)\Big) &= \underline{\dim}_{\Pi,\mathsf{B}}\big(\operatorname{graph}\big(f\big)\big)\,,\\ \overline{\dim}_{\Pi,\mathsf{B}}\Big(\operatorname{graph}\big(p|_X + \lambda f\big)\Big) &= \overline{\dim}_{\Pi,\mathsf{B}}\big(\operatorname{graph}\big(f\big)\big)\,.\end{aligned}$$

(2) *We have*

$$\begin{aligned}\underline{\dim}_{\Pi,\mathsf{B}}\big(\operatorname{graph}\big(p|_X\big)\big) &= \underline{\dim}_{\Pi,\mathsf{B}}(X)\,,\\ \overline{\dim}_{\Pi,\mathsf{B}}\big(\operatorname{graph}\big(p|_X\big)\big) &= \overline{\dim}_{\Pi,\mathsf{B}}(X)\,.\end{aligned}$$

Proof.
(1) It follows from Lemma 5.4 that there are constants $c > 1$ and $C > 1$ such that

$$\frac{1}{C}\,N_{c\delta}\big(\operatorname{graph}(p|_X + \lambda f)\big) \leq N_\delta\big(\operatorname{graph}(f)\big) \leq C\,N_{\frac{\delta}{c}}\big(\operatorname{graph}(p|_X + \lambda f)\big)$$

for all $\delta > 0$, and Lemma 5.1.(1) therefore implies that

$$\frac{1}{C\,(c+2)^d}\,N_\delta\big(\operatorname{graph}(p|_X + \lambda f)\big) \leq N_\delta\big(\operatorname{graph}(f)\big) \leq C\,(c+2)^d\,N_\delta\big(\operatorname{graph}(p|_X + \lambda f)\big)$$

for all $\delta > 0$. We conclude from the above inequality that

$$\begin{aligned}-k\int \frac{1}{s}\,d\Pi_t(s) + &\int \frac{\log N_{e^{-s}}\big(\operatorname{graph}(p|_X + \lambda f)\big)}{s}\,d\Pi_t(s)\\ &\leq \int \frac{\log N_{e^{-s}}\big(\operatorname{graph}(f)\big)}{s}\,d\Pi_t(s)\\ &\leq k\int \frac{1}{s}\,d\Pi_t(s) + \int \frac{\log N_{e^{-s}}\big(\operatorname{graph}(p|_X + \lambda f)\big)}{s}\,d\Pi_t(s)\,,\end{aligned} \tag{5.2}$$

where $k = \log(C(c+2)^d)$. Finally, since $\frac{1}{t} \to 0$ as $t \to \infty$, we conclude from the Consistency Condition (i.e. Condition (iii)) in the definition of an average system that $\int \frac{1}{s}\,d\Pi_t(s) \to 0$ as $t \to \infty$, and the desired result now follows immediately from this and (5.2).
(2) This statement follows from Part (1) by putting $f = 0$ and $\lambda = 1$. $\square$

Lemma 5.6. *Let $\Pi = (\Pi_t)_{t \geq t_0}$ be an averaging system. Fix a bounded subset X of $\mathbb{R}^d$. Let $(r_n)_n$ be a strictly decreasing sequence of positive real numbers with $r_n \to 0$ and $\frac{\log r_n}{\log r_{n+1}} \to 1$. For $t > 0$, let n_t be the unique positive integer such that*

$$r_{n_t+1} \leq e^{-t} < r_{n_t}\,.$$

Then

$$\begin{aligned}\underline{\dim}_{\Pi,\mathsf{B}}(E) &= \liminf_t \int \frac{\log N_{r_{n_s}}(E)}{-\log r_{n_s}}\,d\Pi_t(s)\,,\\ \overline{\dim}_{\Pi,\mathsf{B}}(E) &= \limsup_t \int \frac{\log N_{r_{n_s}}(E)}{-\log r_{n_s}}\,d\Pi_t(s)\,.\end{aligned}$$

Proof.
We first note that it follows from Lemma 5.1 that

$$\begin{aligned}N_{e^{-t}}(E) &\leq \big(\tfrac{r_{n_t}}{e^{-t}} + 2\big)^d\,N_{r_{n_t}}(E) \leq \Big(3\tfrac{r_{n_t}}{r_{n_t+1}}\Big)^d\,N_{r_{n_t}}(E)\,,\\ N_{r_{n_t}}(E) &\leq 2^d N_{r_{n_t+1}}(E) \leq 2^d\Big(\tfrac{e^{-t}}{r_{n_t+1}} + 2\Big)^d\,N_{e^{-t}}(E) \leq \Big(6\tfrac{r_{n_t}}{r_{n_t+1}}\Big)^d\,N_{e^{-t}}(E)\,,\end{aligned}$$

for all $t > 0$. It follows from this that

$$\begin{aligned}
\frac{\log N_{e^{-t}}(E)}{t} &\leq \frac{\log((3\frac{r_{n_t}}{r_{n_t+1}})^d N_{r_{n_t}}(E))}{-\log r_{n_t}} = f(t) + \frac{\log N_{r_{n_t}}(E)}{-\log r_{n_t}}\,,\\
\frac{\log N_{r_{n_t}}(E)}{-\log r_{n_t}} &\leq \frac{\log((6\frac{r_{n_t}}{r_{n_t+1}})^d N_{e^{-t}}(E))}{-\log r_{n_t}} = g(t) + \frac{\log N_{e^{-t}}(E)}{-\log r_{n_t}}\\
&\leq g(t) + \frac{\log r_{n_t+1}}{\log r_{n_t}}\,\frac{\log N_{e^{-t}}(E)}{t} = g(t) + h(t) + \frac{\log N_{e^{-t}}(E)}{t}\,,
\end{aligned}$$

for all $t > 0$, and so

$$\begin{aligned}
-\int f\,d\Pi_t + &\int \frac{\log N_{e^{-s}}(E)}{s}\,d\Pi_t(s)\\
&\leq \int \frac{\log N_{r_{n_s}}(E)}{-\log r_{n_s}}\,d\Pi_t(s)\\
&\leq \int g\,d\Pi_t + \int h\,d\Pi_t + \int \frac{\log N_{e^{-s}}(E)}{s}\,d\Pi_t(s)\,, \qquad (5.3)
\end{aligned}$$

where the functions $f, g, h : (0,\infty) \to [0,\infty)$ are defined by $f(t) = -d(\frac{\log 3}{\log r_{n_t}} + 1 - \frac{\log r_{n_t+1}}{\log r_{n_t}})$, $g(t) = -d(\frac{\log 6}{\log r_{n_t}} + 1 - \frac{\log r_{n_t+1}}{\log r_{n_t}})$ and $h(t) = (\frac{\log r_{n_t+1}}{\log r_{n_t}} - 1)\frac{\log N_{e-t}(E)}{t}$. Since $\frac{\log r_{n+1}}{\log r_n} \to 1$ and $\limsup_t \frac{\log N_{e-t}(E)}{t} = \overline{\dim}_{\mathsf{B}}(E) \leq d$, we conclude that $f(t) \to 0$, $g(t) \to 0$ and $h(t) \to 0$, whence

$$\int f\,d\Pi_t \to 0\,,\quad \int g\,d\Pi_t \to 0\,,\quad \int h\,d\Pi_t \to 0\,. \qquad (5.4)$$

The desired conclusion follows from combining (5.3) and (5.4). □

6. Proof of Theorem 1.1.(2).

The purpose of this section is to prove Theorem 1.1.(2). We first prove three auxiliary lemmas. The first lemma (i.e. Lemma 6.1) is standard and is a suitable version of the reverse Fatou's lemma.

Lemma 6.1. The reverse Fatou's Lemma [St, Theorem 3.2.3]. *Let $(M, \mathcal{E}, \mu)$ be a measure space and let $(\varphi_n)_n$ be a sequence of positive measurable functions $\varphi_n : M \to [0,\infty]$. If $\int \sup_n \varphi_n\,d\mu < \infty$, then $\limsup_n \int \varphi_n\,d\mu \leq \int \limsup_n \varphi_n\,d\mu$.*

Lemma 6.2. *Fix a bounded subset X of $\mathbb{R}^d$. Let $f \in C_{\mathsf{u}}(X)$ and $\delta > 0$. Then there is a positive real number $r > 0$ such that if $g \in B(f, r)$, then*

$$N_\delta\big(\overline{\text{graph}(g)}\big) \leq N_\delta\big(\overline{\text{graph}(f)}\big)\,.$$

Proof.
Let

$$r = \frac{1}{2} \inf_{\substack{Q' \in \mathcal{Q}_\delta^{d+1}\\ Q' \cap \overline{\text{graph}(f)} \neq \varnothing}} \inf_{\substack{Q'' \in \mathcal{Q}_\delta^{d+1}\\ Q'' \cap \overline{\text{graph}(f)} = \varnothing}} \text{dist}(\,Q' \cap \overline{\text{graph}(f)}\,,\,Q''\,)$$

(recall, that for $\delta > 0$, the family $\mathcal{Q}_\delta^{d+1}$ of δ-cubes in $\mathbb{R}^{d+1}$ is defined in (1.2)). First note that since $\overline{\text{graph}(f)}$ is compact, we have $r > 0$. Next, we prove that if $g \in B(f, r)$, then

$$N_\delta\big(\overline{\text{graph}(g)}\big) \leq N_\delta\big(\overline{\text{graph}(f)}\big)\,. \qquad (6.2)$$

Indeed, let Let $g \in B(f, r)$. Since $\|f - g\|_\infty < r$, it follows from the definition of r that

$$\Big\{Q \in \mathcal{Q}_\delta^{d+1}\,\Big|\,Q \cap \overline{\text{graph}(g)} \neq \varnothing\Big\} \subseteq \Big\{Q \in \mathcal{Q}_\delta^{d+1}\,\Big|\,Q \cap \overline{\text{graph}(f)} \neq \varnothing\Big\}\,,$$

whence $N_\delta\big(\,\overline{\text{graph}(g)}\,\big) \leq N_\delta\big(\,\overline{\text{graph}(f)}\,\big)$. This proves (6.2). □

Lemma 6.3. *Let $\Pi = (\Pi_t)_{t\geq t_0}$ be an averaging system. Fix a bounded subset X of $\mathbb{R}^d$. Let $c \in \mathbb{R}$ and $t \geq t_0$. Then the set*

$$\left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \int \frac{\log N_{e^{-s}}\big(\overline{\mathrm{graph}(f)}\big)}{s}\, d\Pi_t(s) < c \right\}$$

is open in $C_{\mathsf{u}}(X)$.

Proof.
Write

$$\begin{aligned} F &= C_{\mathsf{u}}(X) \setminus \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \int \frac{\log N_{e^{-s}}\big(\overline{\mathrm{graph}(f)}\big)}{s}\, d\Pi_t(s) < c \right\} \\ &= \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \int \frac{\log N_{e^{-s}}\big(\overline{\mathrm{graph}(f)}\big)}{s}\, d\Pi_t(s) \geq c \right\}. \end{aligned}$$

We must now prove that F is closed. We therefore fix a sequence $(f_n)_n$ in F and $f \in C_{\mathsf{u}}(X)$ with $\|f_n - f\|_\infty \to 0$. We must now prove that $f \in F$, i.e. we must prove that

$$\int \frac{\log N_{e^{-s}}\big(\overline{\mathrm{graph}(f)}\big)}{s}\, d\Pi_t(s) \geq c\,. \tag{6.3}$$

For brevity define functions $\varphi, \varphi_n : [t_0,\infty) \to [0,\infty)$ by $\varphi(s) = \frac{\log N_{e^{-s}}\big(\overline{\mathrm{graph}(f)}\big)}{s}$ and $\varphi_n(s) = \frac{\log N_{e^{-s}}\big(\overline{\mathrm{graph}(f_n)}\big)}{s}$. We now prove the following three claims.

Claim 1. We have $\int \sup_n \varphi_n \, d\Pi_t < \infty$.
Proof of Claim 1. The measure Π_t has compact support, and we can therefore choose $T_0 \geq t_0$, such that $\operatorname{supp}\Pi_t \subseteq [t_0, T_0]$. Also, the set X is bounded, and we can therefore find a real number a with $X \subseteq [-a,a]^d$. Next, note that since f is bounded and $\|f_n - f\|_\infty \to 0$, there is a constant $M > 0$ such that $|f_n| \leq M$ for all n. From the above we now deduce that $\overline{\mathrm{graph}(f_n)} \subseteq [-a,a]^d \times [-M,M]$ for all n, whence $N_{e^{-s}}(\overline{\mathrm{graph}(f_n)}) \leq N_{e^{-s}}([-a,a]^d \times [-M,M]) \leq (2a+2)^d(2M+2)e^{(d+1)s}$ for all n and all s, and so

$$\varphi_n(s) = \frac{\log N_{e^{-s}}\big(\overline{\mathrm{graph}(f_n)}\big)}{s} \leq \frac{\log((2a+2)^d(2M+2)e^{(d+1)s})}{s} \leq \frac{\log((2a+2)^d(2M+2)e^{(d+1)T_0})}{t_0}$$

for all n and all $s \in [t_0, T_0]$. In particular, since $\operatorname{supp}\Pi_t \subseteq [t_0,T_0]$, we therefore conclude that $\int \sup_n \varphi_n \, d\Pi_t = \int_{t_0}^{T_0} \sup_n \varphi_n \, d\Pi_t \leq \frac{\log((2a+2)^d(2M+2)e^{(d+1)T_0})}{t_0} \Pi_t([t_0,T_0]) < \infty$ This completes the proof of Claim 1.

Claim 2. We have $c \leq \int \limsup_n \varphi_n \, d\Pi_t$.
Proof of Claim 2. Since $f_n \in F$, we conclude that $c \leq \int \frac{\log N_{e^{-s}}(\overline{\mathrm{graph}(f_n)})}{s}\, d\Pi_t(s) = \int \varphi_n \, d\Pi_t$ for all n, whence

$$c \leq \limsup_n \int \varphi_n \, d\Pi_t\,. \tag{6.4}$$

We also note that it follows from Claim 1 and Lemma 6.1 (i.e. the reverse Fatou's Lemma) that

$$\limsup_n \int \varphi_n \, d\Pi_t \leq \int \limsup_n \varphi_n \, d\Pi_t\,. \tag{6.5}$$

The desired result now follows from (6.4) and (6.5). This completes the proof of Claim 2.

Claim 3. For all $s \geq t_0$, we have $\limsup_n \varphi_n(s) \leq \varphi(s)$.

Proof of Claim 3. Fix $s \geq t_0$. We first note that it follows from Lemma 6.2 that there is a positive number $r_s > 0$ such that if $g \in B(f, r_s)$, then $N_{e^{-s}}(\overline{\mathrm{graph}(g)}) \leq N_{e^{-s}}(\overline{\mathrm{graph}(f)})$. Also, since $\|f_n - f\|_\infty \to 0$, there is a positive integer n_s such that $f_n \in B(f, r_s)$ for all $n \geq n_s$. In particular, we conclude that $N_{e^{-s}}(\overline{\mathrm{graph}(f_n)}) \leq N_{e^{-s}}(\overline{\mathrm{graph}(f)})$ for all $n \geq n_s$, and so

$$\varphi_n(s) = \frac{\log N_{e^{-s}}\left(\overline{\mathrm{graph}(f_n)}\right)}{s} \leq \frac{\log N_{e^{-s}}\left(\overline{\mathrm{graph}(f)}\right)}{s} = \varphi(s) \quad \text{for all } n \geq n_s,$$

This clearly implies that $\limsup_n \varphi_n(s) \leq \varphi(s)$. This completes the proof of Claim 3.

Finally, we deduce from Claim 2 and Claim 3 that

$$c \leq \int \limsup_n \varphi_n \, d\Pi_t \leq \int \varphi \, d\Pi_t = \int \frac{\log N_{e^{-s}}\left(\overline{\mathrm{graph}(f)}\right)}{s} \, d\Pi_t(s) \,.$$

This proves (6.3). □

We now turn towards the proof of Theorem 1.1.(2).

Proof of Theorem 1.1.(2).
We must prove that for a typical $f \in C_{\mathsf{u}}(X)$, we have $\underline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) = \underline{\dim}_{\Pi,\mathsf{B}}(X)$. Because of Part (1) in Theorem 1 it clearly suffices to prove that for a typical $f \in C_{\mathsf{u}}(X)$, we have $\underline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) \leq \underline{\dim}_{\Pi,\mathsf{B}}(X)$, i.e. it suffices to prove that the set

$$M = \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \underline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) > \underline{\dim}_{\Pi,\mathsf{B}}(X) \right\}$$

is meagre.

For $u > 0$, write

$$M_u = \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \underline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) > \underline{\dim}_{\Pi,\mathsf{B}}(X) + u \right\}.$$

Since

$$M = \bigcup_{\substack{u \in \mathbb{Q} \\ u > 0}} M_u \,,$$

it suffices to show that M_u is meagre for all $u \in \mathbb{Q}$ with $u > 0$. We therefore fix $u \in \mathbb{Q}$ with $u > 0$. Since $C_{\mathsf{u}}(X)$ is a complete metric space when equipped with the uniform norm, it suffices to show that there is a countable family $(G_n)_n$ of open and dense subsets of $C_{\mathsf{u}}(X)$ with $\cap_n G_n \subseteq C_{\mathsf{u}}(X) \setminus M_u$.

For $t \geq t_0$, let

$$L_t = \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \int \frac{\log N_{e^{-s}}\left(\overline{\mathrm{graph}(f)}\right)}{s} \, d\Pi_t(s) < \underline{\dim}_{\Pi,\mathsf{B}}(X) + u \right\},$$

and for a positive integer n, put

$$G_n = \bigcup_{t \geq n} L_t \,.$$

Below we show that the family $(G_n)_n$ consists of open and dense subsets of $C_{\mathsf{u}}(X)$ with $\cap_n G_n \subseteq C_{\mathsf{u}}(X) \setminus M_u$; this is the contents of the following three claims.

Claim 1. The set G_n is open in $C_{\mathsf{u}}(X)$.
Proof of Claim 1. Indeed, since it follows from Lemma 6.3 that L_t is open for all $t \geq t_0$, we immediately conclude that $G_n = \cup_{t \geq n} L_t$ is open. This completes the proof of Claim 1

Claim 2. The set G_n is dense in $C_{\mathsf{u}}(X)$.

Proof of Claim 2. Let $f \in C_{\mathsf{u}}(X)$ and let $r > 0$. We must now find $g \in C_{\mathsf{u}}(X)$ such that $\|g - f\|_\infty < r$ and $g \in G_n$. We first note that it follows from Lemma 5.3 that there is a polynomial $p : \mathbb{R}^d \to \mathbb{R}$ such that $\|f - p|_X\|_\infty < r$. Put $g = p|_X$. It is clear that g is uniformly continuous and that $\|g - f\|_\infty < r$. We will now prove that $g \in G_n$. It follows from Lemma 5.2 and Lemma 5.5 that

$$
\begin{aligned}
\liminf_{t\to\infty} \int \frac{\log N_{e^{-s}}\big(\overline{\operatorname{graph}(g)}\big)}{s}\, d\Pi_t(s) &= \underline{\dim}_{\Pi,\mathsf{B}}\big(\overline{\operatorname{graph}(g)}\big)\\
&= \underline{\dim}_{\Pi,\mathsf{B}}\big(\operatorname{graph}(g)\big)\\
&= \underline{\dim}_{\Pi,\mathsf{B}}\big(\operatorname{graph}(p|_X)\big)\\
&= \underline{\dim}_{\Pi,\mathsf{B}}(X)\\
&< \underline{\dim}_{\Pi,\mathsf{B}}(X) + u\,.
\end{aligned}
$$

This inequality shows that we can find $t \geq n$ such that $\int \frac{\log N_{e^{-s}}(\overline{\operatorname{graph}(g)})}{s}\, d\Pi_t(s) < \underline{\dim}_{\Pi,\mathsf{B}}(X) + u$, whence $g \in L_t \subseteq G_n$. This completes the proof of Claim 2.

Claim 3. We have $\cap_n G_n \subseteq C_{\mathsf{u}}(X) \setminus M_u$.
Proof of Claim 3. Let $f \in \cap_n G_n$. Hence for each positive integer n, we can find $t_n \geq n$ such that $f \in L_{t_n}$, whence $\int \frac{\log N_{e^{-s}}(\operatorname{graph}(f))}{s}\, d\Pi_{t_n}(s) < \underline{\dim}_{\Pi,\mathsf{B}}(X) + u$ for all positive integers n, and so $\liminf_{t\to\infty} \int \frac{\log N_{e^{-s}}(\operatorname{graph}(f))}{s}\, d\Pi_t(s) \leq \liminf_n \int \frac{\log N_{e^{-s}}(\operatorname{graph}(f))}{s}\, d\Pi_{t_n}(s) \leq \underline{\dim}_{\Pi,\mathsf{B}}(X) + u$. It follows from this that $f \in C_{\mathsf{u}}(X) \setminus M_u$. This completes the proof of Claim 3.

Combining Claim 1, Claim 2 and Claim 3, we now conclude that M_u is meagre. $\square$

7. Proof of Theorem 1.1.(3).(i)

The purpose of this section is to prove Theorem 1.1.(3).(i). We start by providing an alternative characterization of the box dimension (see Lemma 7.1) based on open cubes (as opposed to the usual definition (1.2)-(1.5) based on closed cubes). The motivation for introducing this characterization is the following. Namely, the proof of Theorem 1.1.(3).(i) requires a lower bound for the upper box dimension of the graph of a typical function, and methods for establishing good lower bounds for the box dimension of subsets E of $\mathbb{R}^m$ are often sensitive to the number of cubes from the grid $\mathcal{Q}_\delta^m$ who only intersect E by their boundaries. It is to overcome this problem that we provide an alternative characterization of the box dimension based on open cubes. We first introduce some notation. For $\delta > 0$ and $\mathbf{u} \in \mathbb{R}^m$ write

$$
\mathcal{Q}^{\circ,m}_{\mathbf{u},\delta} = \left\{ \prod_{i=1}^m (n_i\delta, (n_i+1)\delta) \,\middle|\, (n_1, \dots, n_m) \in \mathbf{u} + \mathbb{Z}^m \right\}. \tag{7.1}
$$

Also, for a subset E of $\mathbb{R}^m$, we will write $N^\circ_{\mathbf{u},\delta}(E)$ for the number of open boxes from $\mathcal{Q}^{\circ,m}_{\mathbf{u},\delta}$ that intersect E, i.e.

$$
N^\circ_{\mathbf{u},\delta}(E) = \left| \left\{ Q \in \mathcal{Q}^{\circ,m}_{\mathbf{u},\delta} \,\middle|\, Q \cap E \neq \varnothing \right\} \right|.
$$

Finally, we write

$$
U_m = \left\{ (u_1, \dots, u_m) \,\middle|\, u_i = 0, \tfrac{1}{2} \right\}.
$$

and put

$$
N^\circ_\delta(E) = \sum_{\mathbf{u}\in U_m} N^\circ_{\mathbf{u},\delta}(E)\,. \tag{7.2}
$$

Lemma 7.1. *Fix a bounded subset E of $\mathbb{R}^m$.*

1. *For all $\delta > 0$, we have $N^{\circ}_{\delta}(E) \leq 3^m\, N_{\delta}(E)$.*
2. *For all $\delta > 0$, we have $N_{\delta}(E) \leq 3^m\, N^{\circ}_{\delta}(E)$.*
3. *Let $\Pi = (\Pi_t)_{t \geq t_0}$ be an averaging system. We have*

$$\underline{\dim}_{\Pi,\mathsf{B}}(E) = \liminf_t \int \frac{\log N^{\circ}_{e^{-s}}(E)}{s}\, d\Pi_t(s)\,,$$
$$\overline{\dim}_{\Pi,\mathsf{B}}(E) = \limsup_t \int \frac{\log N^{\circ}_{e^{-s}}(E)}{s}\, d\Pi_t(s)\,.$$

Proof.
(1)-(2) This follows from standard arguments, and for the sake of brevity we have therefore decided to omit the proof.
(3) It follows from (1) and (2) that

$$-k\int f\, d\Pi_t + \int \frac{\log N_{e^{-s}}(E)}{s}\, d\Pi_t(s) \leq \int \frac{\log N^{\circ}_{e^{-s}}(E)}{s}\, d\Pi_t(s) \leq k\int f\, d\Pi_t + \int \frac{\log N_{e^{-s}}(E)}{s}\, d\Pi_t(s)\,, \tag{7.3}$$

where $k = \log(3^m)$ and where the function $f : (0,\infty) \to [0,\infty)$ is defined by $f(t) = \frac{1}{t}$. Since $f(t) \to 0$ as $t \to \infty$, we deduce that $\int f\, d\Pi_t \to 0$ as $t \to \infty$, and the desired result now follows from this and (7.3). $\square$

Lemma 7.2. *Fix a bounded subset X of $\mathbb{R}^d$. Let $f \in C_{\mathsf{u}}(X)$ and $\delta > 0$. Then there is a positive real number $r > 0$ such that if $g \in B(f,r)$, then*

$$N^{\circ}_{\delta}(\mathrm{graph}(f)) \leq N^{\circ}_{\delta}(\mathrm{graph}(g))\,.$$

Proof.
For each $\mathbf{u} = (u_1, \ldots, u_{d+1}) \in U_{d+1}$, write

$$E_{\mathbf{u},\delta} = \bigcup_{m \in u_{d+1}+\mathbb{Z}} \Big(\mathbb{R}^d \times \{m\delta\}\Big)\,,$$

i.e. the $E_{\mathbf{u},\delta}$'s denote the "horizontal" hyperplanes that outline the grid $\mathcal{Q}^{\circ,d+1}_{\mathbf{u},\delta}$. For each $Q \in \mathcal{Q}^{\circ,d+1}_{\mathbf{u},\delta}$ with $Q \cap \mathrm{graph}(f) \neq \varnothing$, choose $x_Q \in Q \cap \mathrm{graph}(f)$ and put

$$r = \frac{1}{2} \min_{\mathbf{u} \in U_{d+1}} \min_{\substack{Q \in \mathcal{Q}^{\circ,d+1}_{\mathbf{u},\delta} \\ Q \cap \mathrm{graph}(f) \neq \varnothing}} \mathrm{dist}(x_Q, E_{\mathbf{u},\delta})\,.$$

We first prove that $r > 0$. Indeed, for all $\mathbf{u} \in U_{d+1}$ and $Q \in \mathcal{Q}^{\circ,d+1}_{\mathbf{u},\delta}$ with $Q \cap \mathrm{graph}(f) \neq \varnothing$ we have $x_Q \in Q \cap \mathrm{graph}(f) \subseteq Q$, whence $x_Q \notin E_{\mathbf{u},\delta}$. We conclude from this that $\mathrm{dist}(x_Q, E_{\mathbf{u},\delta}) > 0$, and so $r > 0$. Next we prove that if $g \in B(f,r)$, then

$$N^{\circ}_{\delta}(\mathrm{graph}(f)) \leq N^{\circ}_{\delta}(\mathrm{graph}(g))\,. \tag{7.4}$$

Indeed, let $g \in B(f,r)$. Since $\|f - g\|_{\infty} < r$, the definition of r implies that if $\mathbf{u} \in U_{d+1}$, then

$$\Big\{Q \in \mathcal{Q}^{\circ,d+1}_{\mathbf{u},\delta} \,\Big|\, Q \cap \mathrm{graph}(f) \neq \varnothing\Big\} \subseteq \Big\{Q \in \mathcal{Q}^{\circ,d+1}_{\mathbf{u},\delta} \,\Big|\, Q \cap \mathrm{graph}(g) \neq \varnothing\Big\}\,.$$

This clearly implies that $N^{\circ}_{\mathbf{u},\delta}(\mathrm{graph}(f)) \leq N^{\circ}_{\mathbf{u},\delta}(\mathrm{graph}(g))$, and so $N^{\circ}_{\delta}(\mathrm{graph}(f)) \leq N^{\circ}_{\delta}(\mathrm{graph}(g))$. $\square$

Lemma 7.3. *Let $\Pi = (\Pi_t)_{t\geq t_0}$ be an averaging system. Fix a bounded subset X of $\mathbb{R}^d$. Let $c \in \mathbb{R}$ and $t \geq t_0$. Then the set*

$$\left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \int \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(f))}{s}\, d\Pi_t(s) > c \right\}$$

is open in $C_{\mathsf{u}}(X)$.

Proof.
Write

$$\begin{aligned} F &= C_{\mathsf{u}}(X) \setminus \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \int \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(f))}{s}\, d\Pi_t(s) > c \right\} \\ &= \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \int \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(f))}{s}\, d\Pi_t(s) \leq c \right\}. \end{aligned}$$

We must now prove that F is closed. We therefore fix a sequence $(f_n)_n$ in F and $f \in C_{\mathsf{u}}(X)$ with $\|f_n - f\|_\infty \to 0$. We must now prove that $f \in F$, i.e. we must prove that

$$\int \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(f))}{s}\, d\Pi_t(s) \leq c\,. \tag{7.5}$$

For brevity define functions $\varphi, \varphi_n : [t_0, \infty) \to [0, \infty)$ by $\varphi(s) = \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(f))}{s}$ and $\varphi_n(s) = \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(f_n))}{s}$. We now prove the following two claims.

Claim 1. We have $\int \liminf_n \varphi_n \, d\Pi_t \leq c$.
Proof of Claim 1. Since $f_n \in F$, we conclude that $\int \varphi_n \, d\Pi_t = \int \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(f_n))}{s}\, d\Pi_t(s) \leq c$ for all n, whence

$$\liminf_n \int \varphi_n \, d\Pi_t \leq c\,. \tag{7.6}$$

We also note that it follows from Fatou's lemma that

$$\int \liminf_n \varphi_n \, d\Pi_t \leq \liminf_n \int \varphi_n \, d\Pi_t\,. \tag{7.7}$$

The desired result now follows from (7.6) and (7.7). This competes the proof of Claim 2.

Claim 2. For all $s \geq t_0$, we have $\varphi(s) \leq \liminf_n \varphi_n(s)$.
Proof of Claim 2. Fix $s \geq t_0$. We first note that it follows from Lemma 7.2 that there is a positive number $r_s > 0$ such that if $g \in B(f, r_s)$, then $N^{\circ}_{e^{-s}}(\mathrm{graph}(f)) \leq N^{\circ}_{e^{-s}}(\mathrm{graph}(g))$. Also, since $\|f_n - f\|_\infty \to 0$, there is a positive integer n_s such that $f_n \in B(f, r_s)$ for all $n \geq n_s$. In particular, we conclude that $N^{\circ}_{e^{-s}}(\mathrm{graph}(f)) \leq N^{\circ}_{e^{-s}}(\mathrm{graph}(f_n))$ for all $n \geq n_s$, and so

$$\varphi(s) = \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(f))}{s} \leq \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(f_n))}{s} = \varphi_n(s) \quad \text{for all } n \geq n_s,$$

This clearly implies that $\varphi(s) \leq \liminf_n \varphi_n(s)$. This completes the proof of Claim 2.

Finally, we deduce from Claim 1 and Claim 2 that

$$\int \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(f))}{s}\, d\Pi_t(s) = \int \varphi \, d\Pi_t \leq \int \liminf_n \varphi_n \, d\Pi_t \leq c\,.$$

This proves (7.5). □

We now turn towards the proof of Theorem1.1.(3).(i).

Proof of Theorem 1.1.(3).(i).
For brevity write $A = \sup_{f\in C_{\mathsf{u}}(X)} \overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f))$. We must prove that for a typical $f \in C_{\mathsf{u}}(X)$, we have $\overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) = A$. It clearly suffices to prove that for a typical $f \in C_{\mathsf{u}}(X)$, we have $\overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) \geq A$, i.e. it suffices to prove that the set

$$M = \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) < A \right\}$$

is meagre.

For $u > 0$, write

$$M_u = \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(f)) < A - u \right\}.$$

Since

$$M = \bigcup_{\substack{u\in\mathbb{Q}\\ u>0}} M_u\,,$$

it clearly suffices to show that M_u is meagre for all $u \in \mathbb{Q}$ with $u > 0$. We therefore fix $u \in \mathbb{Q}$ with $u > 0$. Since $C_{\mathsf{u}}(X)$ is a complete metric space when equipped with the uniform norm, it suffices to show that there is a countable family $(G_n)_n$ of open and dense subsets of $C_{\mathsf{u}}(X)$ with $\cap_n G_n \subseteq C_{\mathsf{u}}(X) \setminus M_u$.

For $t \geq t_0$, let

$$L_t = \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \int \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(f))}{s}\, d\Pi_t(s) > A - u \right\},$$

and for a positive integer n, put

$$G_n = \bigcup_{t\geq n} L_t\,.$$

Below we show that the family $(G_n)_n$ consists of open and dense subsets of $C_{\mathsf{u}}(X)$ with $\cap_n G_n \subseteq C_{\mathsf{u}}(X) \setminus M_u$; this is the contents of the following three claims.

Claim 1. The set G_n is open in $C_{\mathsf{u}}(X)$.
Proof of Claim 1. Indeed, since it follows from Lemma 7.3 that L_t is open for all $t \geq t_0$, we immediately conclude that $G_n = \cup_{t\geq n} L_t$ is open. This completes the proof of Claim 1

Claim 2. The set L_n is dense in $C_{\mathsf{u}}(X)$.
Proof of Claim 2. Let $f \in C_{\mathsf{u}}(X)$ and let $r > 0$. We must now find $g \in G_n$ such that $\|g - f\|_\infty < r$.

Without loss of generality, we may assume $\frac{r}{2} \leq u$. We first note that it follows from Lemma 5.3 that there is a polynomial $p : \mathbb{R}^d \to \mathbb{R}$ such that $\|f - p|_X\|_\infty < r$. We also note that the definition of A implies that there is a function $\varphi \in C_{\mathsf{u}}(X)$ such that

$$\overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(\varphi)) > A - \frac{r}{4}\,. \tag{7.8}$$

Finally, put $c = \frac{r}{4(\|\varphi\|_\infty+1)} > 0$, and define $g : X \to \mathbb{R}$ by $g = p|_X + c\varphi$. Clearly $g \in C_{\mathsf{u}}(X)$ and $\|f - g\|_\infty = \|f - p|_X - c\varphi\|_\infty \leq \|f - p|_X\|_\infty + c\|\varphi\|_\infty = \|F|_X - p|_X\|_\infty + c\|\varphi\|_\infty \leq \frac{r}{4} + c\|\varphi\|_\infty \leq \frac{r}{4} + \frac{r}{4(\|\varphi\|_\infty+1)}\|\varphi\|_\infty < r$. Next, we show that $g \in G_n$. We first note that it follows from Lemma 7.1 that $\limsup_{t\to\infty} \int \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(f))}{s}\, d\Pi_t(s) = \overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(g))$, and we can therefore choose $t \geq n$ such that

$$\int \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(g))}{s}\, d\Pi_t(s) > \overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(g)) - \frac{r}{4}\,. \tag{7.9}$$

Since $\frac{r}{2} \leq u$, we conclude from (7.9) that

$$\begin{aligned}\int \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(g))}{s}\, d\Pi_t(s) + u &\geq \int \frac{\log N^{\circ}_{e^{-s}}(\mathrm{graph}(g))}{s}\, d\Pi_t(s) + \frac{r}{2}\\ &\geq \overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(g)) + \frac{r}{4}\\ &= \overline{\dim}_{\Pi,\mathsf{B}}(\mathrm{graph}(p|_X + c\varphi)) + \frac{r}{4}\,. \end{aligned} \tag{7.10}$$

Also, observe that it follows from Lemma 5.5 that $\overline{\dim}_{\Pi,\mathsf{B}}(\text{graph}(p|_X + c\varphi)) = \overline{\dim}_{\Pi,\mathsf{B}}(\text{graph}(\varphi))$, and we therefore conclude from (7.10) that

$$\int \frac{\log N^{\circ}_{e^{-s}}(\text{graph}(g))}{s}\, d\Pi_t(s) + u \geq \overline{\dim}_{\Pi,\mathsf{B}}(\text{graph}(\varphi)) + \frac{r}{4}\,. \tag{7.11}$$

Finally, combining (7.8) and (7.11) yields

$$\int \frac{\log N^{\circ}_{e^{-s}}(\text{graph}(g))}{s}\, d\Pi_t(s) + u > A$$

This shows that $g \in L_t \subseteq G_n$. This completes the proof of Claim 2.

Claim 3. We have $\cap_n G_n \subseteq C_{\mathsf{u}}(X) \setminus M_u$.
Proof of Claim 3. Let $f \in \cap_n G_n$. Hence for each positive integer n, we can find a real number $t_n \geq n$ such that $f \in L_{t_n}$, whence $\int \frac{\log N^{\circ}_{e^{-s}}(\text{graph}(f))}{s}\, d\Pi_{t_n}(s) > A - u$ for all positive integers n, and so $\limsup_{t\to\infty} \int \frac{\log N^{\circ}_{e^{-s}}(\text{graph}(f))}{s}\, d\Pi_t(s) \geq \limsup_n \int \frac{\log N^{\circ}_{e^{-s}}(\text{graph}(f))}{s}\, d\Pi_{t_n}(s) \geq A - u$. It follows from this that $f \in C_{\mathsf{u}}(X) \setminus M_u$. This completes the proof of Claim 3.

Combining Claim 1, Claim 2 and Claim 3, we now conclude that M_u is meagre. □

8. Proof of Theorem 1.1.(3).(ii).

The purpose of this section is to prove Theorem 1.1.(3).(ii).

Proposition 8.1. *Let X be a bounded subset of $\mathbb{R}^d$ with only finitely many isolated points. Let $\varepsilon > 0$ Then there is a function $f \in C_{\mathsf{u}}(X)$ such that*

$$N_{2^{-n}}(\text{graph}(f)) \geq 2^{-d}\, N_{2^{-n}}(X)\, 2^{n(1-\varepsilon)}$$

for all positive integers n.

Proof.
Observe that if a set has finitely many isolated points, we may remove these without changing the lower and the upper box dimensions of the set. Hence we may suppose that X has no isolated points.

Fix a positive integer n and write

$$\mathcal{V}_n = \Big\{ Q \in \mathcal{Q}^d_{2^{-n}} \,\Big|\, Q \cap X \neq \varnothing \Big\}$$

(recall, that for $\delta > 0$, the family $\mathcal{Q}^d_\delta$ of δ-cubes in $\mathbb{R}^d$ is defined in (1.2)). Since X does not have isolated points there is a subfamily $\mathcal{W}_n$ of $\mathcal{V}_n$ with $|\mathcal{W}_n| \geq \frac{1}{2^d}|\mathcal{V}_n|$ such that if $Q \in \mathcal{W}_n$, then none of the points in the set $X \cap Q$ are isolated in $X \cap Q$.

For each integer n with $n \geq 0$, we will now define a uniformly continuous function $f_n : X \to [0, \infty)$ and a finite set

$$E_n = \Big\{ x_{Q,n} \,\Big|\, Q \in \mathcal{W}_n \Big\} \;\cup\; \Big\{ y_{Q,n,i} \,\Big|\, Q \in \mathcal{W}_n\,,\; i = 1, \ldots, \lceil 2^{n(1-\varepsilon)} \rceil \Big\}$$

such that the following properties are satisfied

$$x_{Q,n}, y_{Q,n,i} \in X \cap Q\,, \tag{8.1}$$

$$\left| \sum_{j=0}^{n-1} f_j(x_{Q,n}) - \sum_{j=0}^{n-1} f_j(y_{Q,n,i}) \right| \leq 2^{-n}\,, \tag{8.2}$$

$$\|f_n\|_\infty \le 5\left\lceil 2^{n(1-\varepsilon)}\right\rceil 2^{-n}\,, \tag{8.3}$$
$$f_n(x_{Q,n}) = 0\,, \tag{8.4}$$
$$f_n(y_{Q,n,i}) = 5i2^{-n}\,, \tag{8.5}$$
$$f_k(y_{Q,n,i}) = 0 \text{ for } k < n\,. \tag{8.6}$$

Below we construct the functions f_n and the sets E_n inductively as follows.

First we put $f_0 = 0$ and $E_0 = \varnothing$. Next assume that the functions $f_0, f_1, \dots, f_{n-1}$ and the sets $E_0, E_1, \dots, E_{n-1}$ have been constructed such that properties (8.1)–(8.6) are satisfied. We will now construct f_n and E_n. Fix $Q \in \mathcal{W}_n$. It follows from the definition of $\mathcal{W}_n$ that we can choose $x_{Q,n} \in (Q \cap X) \setminus (E_0 \cup E_1 \cup \ldots \cup E_{n-1})$. It also follows from the definition of $\mathcal{W}_n$ and the fact that the functions $f_0, f_1, \dots, f_{n-1}$ are (uniformly) continuous that we can choose points $y_{Q,n,i} \in (Q \cap X) \setminus (E_0 \cup E_1 \cup \ldots \cup E_{n-1})$ with $i = 1, \dots, \left\lceil 2^{n(1-\varepsilon)}\right\rceil$ such that the points $x_{Q,n}, y_{Q,n,1}, \dots, y_{Q,n,\lceil 2^{n(1-\varepsilon)}\rceil}$ are distinct and

$$\left|\sum_{j=0}^{n-1} f_j(x_{Q,n}) - \sum_{j=0}^{n-1} f_j(y_{Q,n,i})\right| \le 2^{-n}$$

Now define $g_n : E_0 \cup E_1 \cup \ldots \cup E_{n-1} \cup E_n \to \mathbb{R}$ by

$$g_n(x) = \begin{cases} 0 & \text{if } x \in E_0 \cup E_1 \cup \ldots \cup E_{n-1}; \\ 0 & \text{if } x = x_{Q,n}; \\ 5i2^{-n} & \text{if } x = y_{Q,n,i} \text{ for } i = 1, \dots, \left\lceil 2^{n(1-\varepsilon)}\right\rceil. \end{cases}$$

Next, observe that since the set $E_0 \cup E_1 \cup \ldots \cup E_{n-1} \cup E_n$ is finite, we can find a uniformly continuous function $f_n : X \to [0,\infty)$ such that $f_n|_{E_0 \cup E_1 \cup \ldots \cup E_{n-1} \cup E_n} = g_n$ and $0 = \min_{x \in E_0 \cup E_1 \cup \ldots \cup E_{n-1} \cup E_n} g_n(x) \le f(x) \le \max_{x \in E_0 \cup E_1 \cup \ldots \cup E_{n-1} \cup E_n} g_n(x) = 5\left\lceil 2^{n(1-\varepsilon)}\right\rceil 2^{-n}$ for all $x \in X$. It is clear that the function f_n and the set $E_n = \{x_{Q,n} \,|\, Q \in \mathcal{W}_n\} \,\cup\, \{y_{Q,n,i} \,|\, Q \in \mathcal{W}_n\,,\, i = 1, \dots, \left\lceil 2^{n(1-\varepsilon)}\right\rceil\}$ satisfy the properties in (8.1)–(8.6). This completes the construction of the functions f_n and the sets E_n.

We now construct $f \in C_{\mathsf{u}}(K)$ as follows. Namely, note that it follows from (8.3) that

$$\begin{aligned} \sum_n \|f_n\|_\infty &\le \sum_n 5\left\lceil 2^{n(1-\varepsilon)}\right\rceil 2^{-n} \\ &\le 5\sum_n \left(2^{-n\varepsilon} + 2^{-n}\right) \\ &< \infty\,. \end{aligned} \tag{8.7}$$

We conclude from (8.7) that the function f defined by

$$f = \sum_n f_n$$

is a well-defined real valued uniformly continuous function.

Below we prove that

$$N_{2^{-n}}(\operatorname{graph}(f)) \ge 2^{-d}\, N_{2^{-n}}(X)\, 2^{n(1-\varepsilon)}$$

for all n. This is done in the following 2 claims.

Claim 1. If n is a positive integer and $Q \in \mathcal{W}_n$, then $N_{2^{-n}}(\operatorname{graph}(f|_{Q\cap X})) \ge 2^{n(1-\varepsilon)}$.

Proof of Claim 1. We first show that if $i, j = 1, \dots, \left\lceil 2^{n(1-\varepsilon)}\right\rceil$ with $i \ne j$, then

$$|f(y_{Q,n,i}) - f(y_{Q,n,j})| > 2^{-n}\,. \tag{8.8}$$

Indeed, we have

$$\begin{aligned} &|f_n(y_{Q,n,i}) - f_n(y_{Q,n,j})| \\ &\quad= \left|\left(\sum_{k=0}^{n} f_k(y_{Q,n,i}) - \sum_{k=0}^{n} f_k(y_{Q,n,j})\right) - \left(\sum_{k=0}^{n-1} f_k(y_{Q,n,i}) - \sum_{k=0}^{n-1} f_k(y_{Q,n,j})\right)\right| \\ &\quad\le \left|\sum_{k=0}^{n} f_k(y_{Q,n,i}) - \sum_{k=0}^{n} f_k(y_{Q,n,j})\right| \\ &\qquad+ \left|\sum_{k=0}^{n-1} f_k(y_{Q,n,i}) - \sum_{k=0}^{n-1} f_k(x_{Q,n})\right| + \left|\sum_{k=0}^{n-1} f_k(x_{Q,n}) - \sum_{k=0}^{n-1} f_k(y_{Q,n,j})\right|, \end{aligned}$$

whence

$$
\begin{aligned}
&|f(y_{Q,n,i}) - f(y_{Q,n,j})| \\
&\qquad = \left| \sum_{k=0}^{n} f_k(y_{Q,n,i}) - \sum_{k=0}^{n} f_k(y_{Q,n,j}) \right| && \text{[by (8.6)]} \\
&\qquad \geq |f_n(y_{Q,n,i}) - f_n(y_{Q,n,j})| \\
&\qquad\qquad - \left| \sum_{k=0}^{n-1} f_k(y_{Q,n,i}) - \sum_{k=0}^{n-1} f_k(x_{Q,n}) \right| - \left| \sum_{k=0}^{n-1} f_k(x_{Q,n}) - \sum_{k=0}^{n-1} f_k(y_{Q,n,j}) \right| \\
&\qquad \geq |5i2^{-n} - 5j2^{-n}| - 2^{-n} - 2^{-n} && \text{[by (8.2) and (8.5)]} \\
&\qquad = 5|i-j|2^{-n} - 2^{-n} - 2^{-n} \\
&\qquad \geq 5 \cdot 2^{-n} - 2^{-n} - 2^{-n} \\
&\qquad > 2^{-n} \,.
\end{aligned}
$$

This completes the proof of (8.8).

It follows from (8.8) that distinct points in the set $\{(y_{Q,n,i}, f(y_{Q,n,i}))\,|\, i = 1, \dots, \lceil 2^{n(1-\varepsilon)} \rceil\}$ are at most 2^{-n} close, whence

$$
N_{2^{-n}}(\operatorname{graph}(f|_{Q\cap X})) \geq \left| \{(y_{Q,n,i}, f(y_{Q,n,i}))\,|\, i = 1, \dots, \lceil 2^{n(1-\varepsilon)} \rceil\} \right| = \lceil 2^{n(1-\varepsilon)} \rceil \geq 2^{n(1-\varepsilon)} \,.
$$

This completes the proof of Claim 1.

Claim 2. If n is a positive integer, then $N_{2^{-n}}(\operatorname{graph}(f)) \geq \frac{1}{2^d} N_{2^{-n}}(X)\, 2^{n(1-\varepsilon)}$.
Proof of Claim 2. It follows from Claim 1 that

$$
\begin{aligned}
N_{2^{-n}}(\operatorname{graph}(f)) &= \sum_{Q\in\mathcal{V}_n} N_{2^{-n}}(\operatorname{graph}(f|_{Q\cap X})) \\
&\geq \sum_{Q\in\mathcal{W}_n} N_{2^{-n}}(\operatorname{graph}(f|_{Q\cap X})) \\
&\geq \sum_{Q\in\mathcal{W}_n} 2^{n(1-\varepsilon)} \\
&= |\mathcal{W}_n|\, 2^{n(1-\varepsilon)} \\
&\geq \frac{1}{2^d}\, |\mathcal{V}_n|\, 2^{n(1-\varepsilon)} \\
&= \frac{1}{2^d}\, N_{2^{-n}}(X)\, 2^{n(1-\varepsilon)} \,.
\end{aligned}
$$

This completes the proof of Claim 2.

The desired result follows immediately from Claim 2. □

We can now prove Theorem 1.1.(3).(ii).

Proof of Theorem 1.1.(3).(ii).
For brevity write $A = \sup_{f\in C_{\mathsf{u}}(X)} \overline{\dim}_{\Pi,\mathsf{B}}(\operatorname{graph}(f))$. We must now prove that if X does not have any isolated points, then

$$
A = \overline{\dim}_{\Pi,\mathsf{B}}(X) + 1 \,.
$$

Indeed, it is not difficult to see that $A = \sup_{f\in C_{\mathsf{u}}(X)} \overline{\dim}_{\Pi,\mathsf{B}}(\operatorname{graph}(f)) \leq \overline{\dim}_{\Pi,\mathsf{B}}(X) + 1$. Hence, it suffices to show that $\overline{\dim}_{\Pi,\mathsf{B}}(X) + 1 \leq A$. Let $\varepsilon > 0$, and note that it follows from Proposition 8.1 that we can find a function $f \in C_{\mathsf{u}}(X)$ such that

$$
N_{2^{-n}}(\operatorname{graph}(f)) \geq 2^{-d}\, N_{2^{-n}}(X)\, 2^{n(1-\varepsilon)} \tag{8.9}
$$

for all positive integers n. For $t>0$, let n_t denote the unique positive integer such that

$$2^{-(n_t+1)} \le e^{-t} < 2^{-n_t}\,.$$

We now conclude from inequality (8.9) and Lemma 5.6 (applied to the sequence $(r_n)_n$ defined by $r_n = 2^{-n}$) that

$$\begin{aligned}A &\ge \overline{\dim}_{\Pi,\mathsf{B}}(\operatorname{graph}(f))\\ &= \limsup_t \int \frac{\log N_{2^{n_s}}(\operatorname{graph}(f))}{-\log 2^{-n_s}}\,d\Pi_t(s)\\ &\ge \limsup_t \int \frac{\log(2^{-d}\,N_{2^{-n_s}}(X)\,2^{n_s(1-\varepsilon)})}{-\log 2^{-n_s}}\,d\Pi_t(s)\\ &= \limsup_t \left(\int h\,d\Pi_t + \int \frac{\log N_{2^{-n_s}}(X)}{-\log 2^{-n_s}}\,d\Pi_t(s)\right),\end{aligned} \tag{8.10}$$

where the function $h:(0,\infty)\to(0,\infty)$ is defined by $h(t) = 1-\varepsilon-\frac{d}{n_t}$. Since $h(t)\to 1-\varepsilon$ as $t\to\infty$, we conclude that $\int h\,d\Pi_t \to 1-\varepsilon$ as $t\to\infty$, and it therefore follows from (8.10) and Lemma 5.6 (once more applied to the sequence $(r_n)_n$ defined by $r_n = 2^{-n}$) that

$$\begin{aligned}A &\ge 1-\varepsilon+\limsup_t \int \frac{\log N_{2^{-n_s}}(X)}{-\log 2^{-n_s}}\,d\Pi_t(s)\\ &= 1-\varepsilon+\overline{\dim}_{\Pi,\mathsf{B}}(X)\,.\end{aligned}$$

Finally, letting $\varepsilon \searrow 0$ gives the desired result. $\square$

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